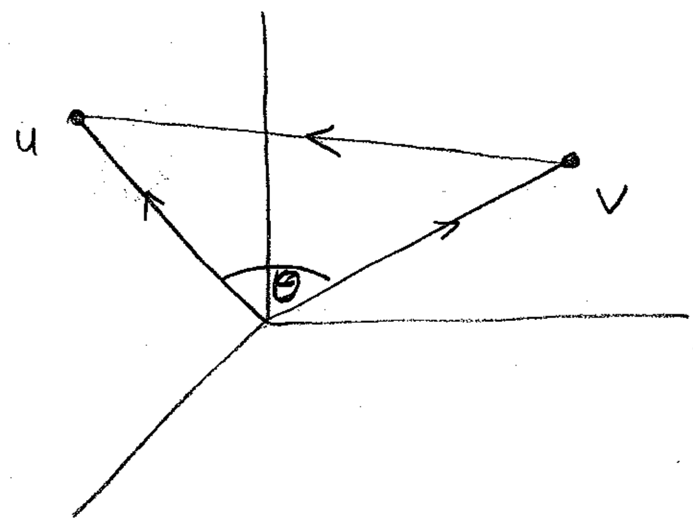
Angles

Consider two non-zero vectors u, v in $\mathbb{R}^n$



Together with $u-v$, they make up a triangle.

Let θ be the angle between u & v.

By the cosine rule

$$\|u-v\|^2 = \|u\|^2 + \|v\|^2 - 2\|u\|\|v\| \cos\theta$$

But

$$\|u-v\|^2 = (u-v)\cdot(u-v) = \|u\|^2 + \|v\|^2 - 2u\cdot v$$

Equating the two expressions for $\|u-v\|^2$ gives

$$-2\|u\|\|v\|\cos\theta = -2\, u\cdot v$$

$$\boxed{\cos\theta = \frac{u\cdot v}{\|u\|\|v\|}}$$

Note that if $u \perp v$, $\cos\theta = \cos\frac{\pi}{2} = 0$ which is what we would expect as $u\cdot v = 0$ in this case.

<u>Example</u> Find the angle between

$u = \begin{bmatrix}1\\1\\2\end{bmatrix}$, $v = \begin{bmatrix}1\\0\\1\end{bmatrix}$ in $\mathbb{R}^3$.

$$\cos\theta = \frac{u\cdot v}{\|u\|\|v\|} = \frac{1\cdot 1 + 1\cdot 0 + 2\cdot 1}{\sqrt{1^2+1^2+2^2}\cdot\sqrt{1^2+1^2}} = \frac{3}{\sqrt{6}\sqrt{2}} = \frac{\sqrt{3}}{2}$$

Thus $\theta = \cos^{-1}\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{6}$ $(= 30^\circ)$