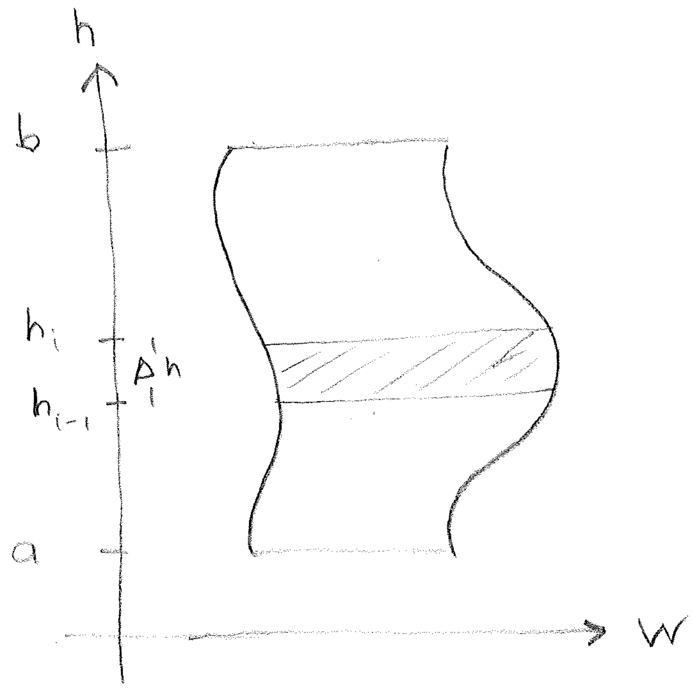
Chapter 8

Using the Definite Integral

§ 8.1 Areas and Volumes

Areas by horizontal slices

Consider the following figure where the width $w = w(h)$ of the horizontal cross-section depends on the height h.



Small horizontal piece is approximately a rectangle of area

$$w_{i-1}\,\Delta h$$

$$= w(h_{i-1})\,\Delta h.$$

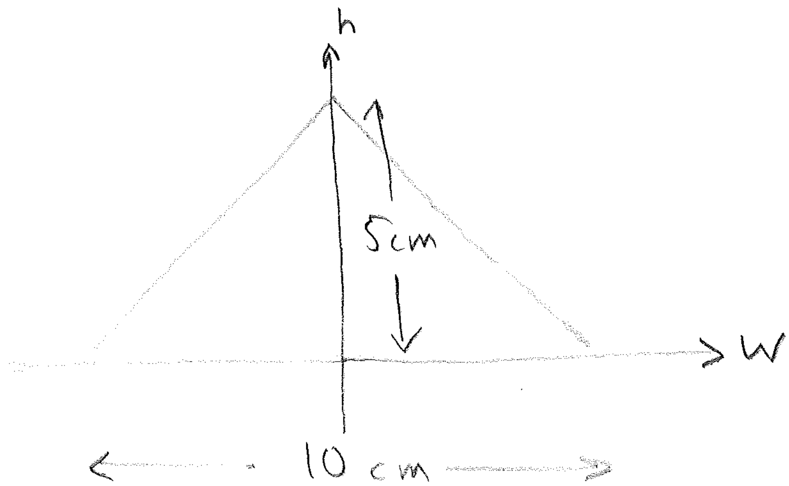
Add all these pieces together to get an approx. for the total area A.

$$A \approx \sum_{i=1}^{n} w(h_{i-1}) \Delta h$$

This is a left-hand Riemann sum for the integral $\int_a^b w(h)\, dh$ and taking limits as $n \to \infty$ we define

$$A = \int_a^b w(h)\, dh.$$

Ex. Find the area of the isoceles triangle below by horizontal slices

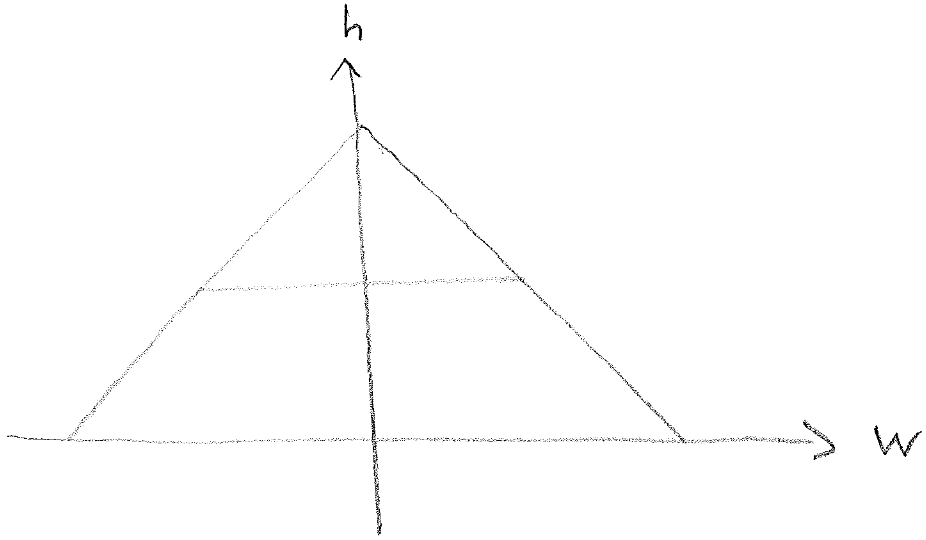


Note that by geometry

$$A = \frac{1}{2} \text{ base} \times \text{height}$$

$$= \frac{1}{2} 10 \times 5$$

$$= 25 \text{ cm}^2.$$



Here w is a linear f^n of h

where $w(0) = 10$, $w(5) = 0$.

Get that $w(h) = 10 - 2h$

$$A = \int_0^5 w(h)\,dh$$

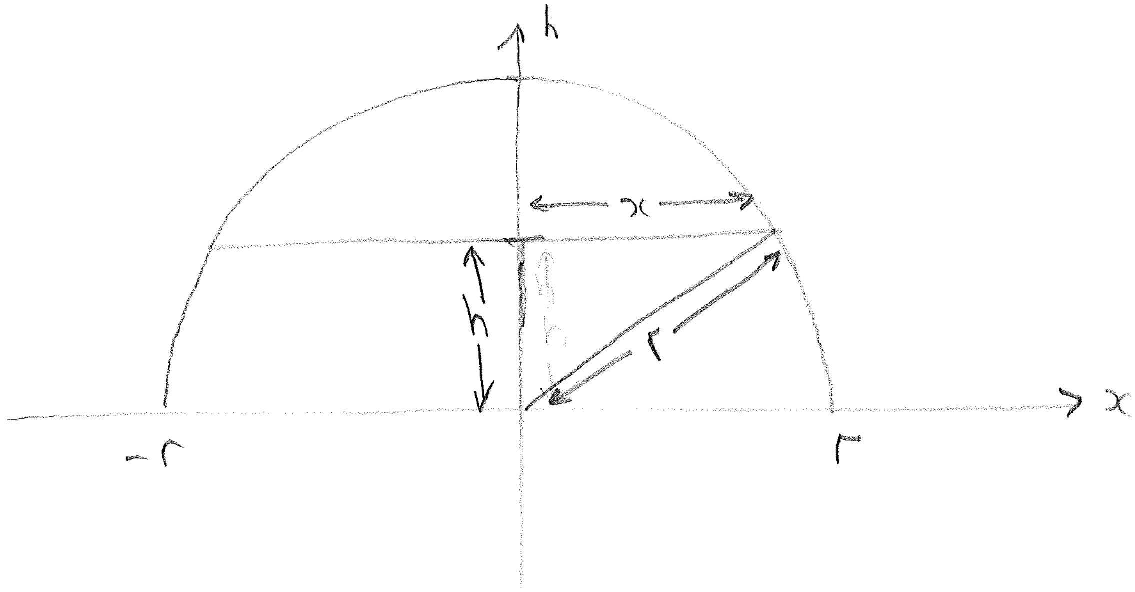
$$= \int_0^5 (10 - 2h)\,dh$$

$$= \left[10h - h^2\right]_0^5$$

$$= 10(5) - 5^2 - 0$$

$$= 25\text{ cm}^2 \text{ (again).}$$

Ex. Semicircle of radius r.



Eq^n of semicircle is $h^2 + x^2 = r^2$

so

$$x^2 = r^2 - h^2 \qquad (0 \leq x \leq r).$$

$$x = \sqrt{r^2 - h^2}$$

Then $\quad w(h) = 2x \quad$ (by symmetry)

$$= 2\sqrt{r^2 - h^2}$$

Thus

$$A = \int_0^r 2\sqrt{r^2 - h^2}\, dh$$

Trig substitution: let $h = r\sin\theta$

$dh = r\cos\theta\, d\theta$

$$\sqrt{r^2 - h^2} = \sqrt{r^2 - r^2\sin^2\theta}$$

$$= r\sqrt{1-\sin^2\theta}$$

$$= r\sqrt{\cos^2\theta}$$

$$= r\cos\theta$$

<u>Limits</u>: when $h = 0$, $\theta = 0$

$h = r$, $\theta = \frac{\pi}{2}$.

So

$$A = \int_0^r 2\sqrt{r^2-h^2}\,dh = \int_0^{\frac{\pi}{2}} 2r^2\cos^2\theta\, d\theta$$

$$= 2r^2 \int_0^{\frac{\pi}{2}} \cos^2\theta\, d\theta$$

$$= 2r^2 \int_0^{\frac{\pi}{2}} \frac{1}{2}(1+\cos 2\theta)\, d\theta$$

as $\cos^2\theta = \frac{1}{2}(1+\cos 2\theta)$

$$= r^2 \int_0^{\frac{\pi}{2}} (1 + \cos 2\theta)\, d\theta$$

$$= r^2 \left[\theta + \frac{\sin 2\theta}{2} \right]_0^{\frac{\pi}{2}}$$

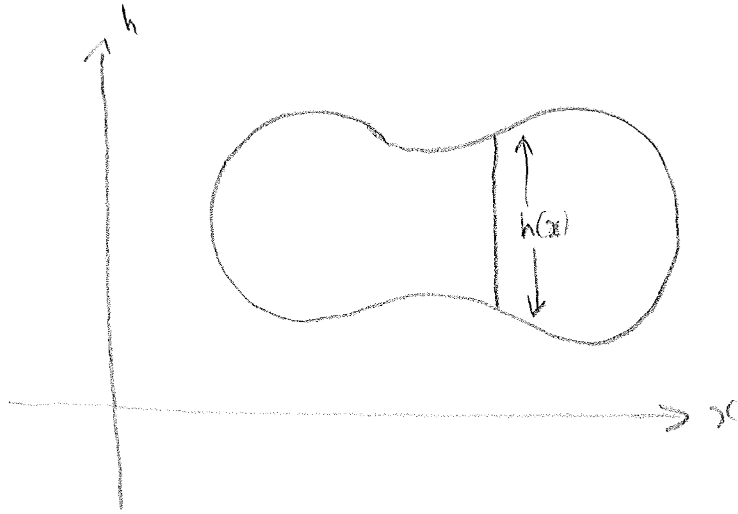
$$= r^2 \left(\left(\frac{\pi}{2} + \frac{\sin \pi}{2} \right) - \left(0 + \frac{\sin 0}{2} \right) \right)$$

$$= \frac{\pi r^2}{2}$$

So the area of the semicircle is $\frac{\pi r^2}{2}$ as we'd expect.

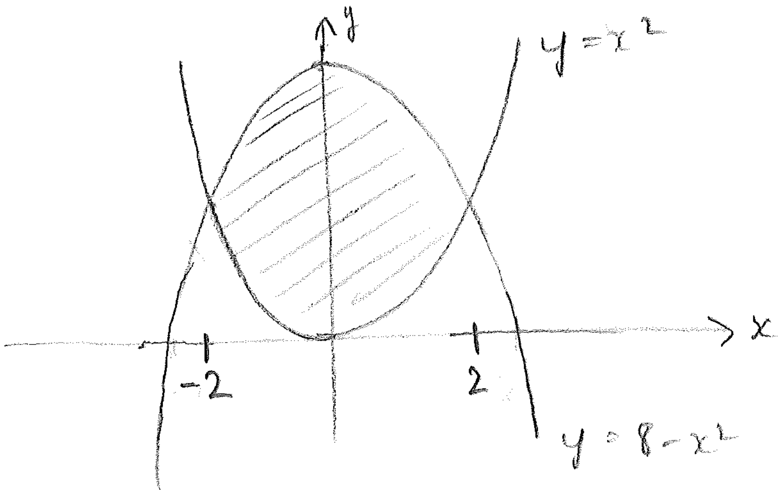
Area by Vertical Cross-Sections

If we slice the region vertically instead of horizontally into slices of height $h(x)$, we get a similar formula



$$A = \int_a^b h(x)\,dx.$$

Ex. Find the area of the region enclosed between the parabolas $y = x^2$ and $y = 8 - x^2$.



Need to intersect the curves to determine the limits of the integration in x.

Set
$$x^2 = 8 - x^2$$
$$2x^2 = 8$$
$$x^2 = 4$$
$$x = \pm 2$$

Integral runs from -2 to 2.

Note that on $[-2, 2]$, $8 - x^2$ is the larger curve. e.g. $8 - 0^2 = 8 > 0 = 0^2$.

Then $h(x) = 8 - x^2 - x^2 = 8 - 2x^2$

and

$$A = \int_{-2}^{2} (8 - 2x^2)\, dx$$

$$= \left[8x - \frac{2x^3}{3} \right]_{-2}^{2}$$

$$= \left(8(2) - \frac{2(8)}{3}\right) - \left(8(-2) - \frac{2(-2)^3}{3}\right)$$

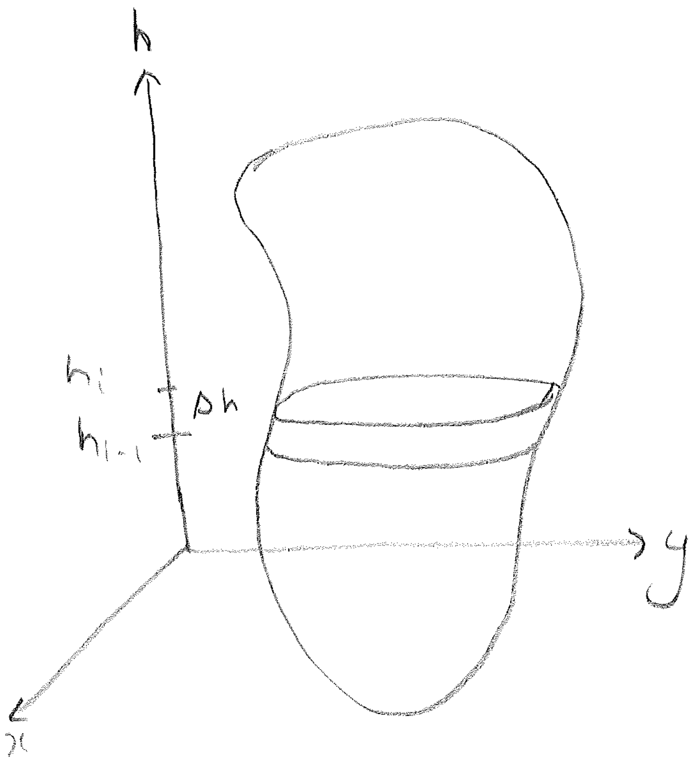
$$= 16 - \frac{16}{3} - \left(-16 + \frac{16}{3}\right)$$

$$= 32 - \frac{32}{3}$$

$$= \frac{64}{3}$$

Volumes by horizontal slices

Consider the figure below where the area $A = A(h)$ of the horizontal cross-section depends on the height h



Small horizontal piece is approx. a slice of volume

$$A(h_{i-1})\,\Delta h$$

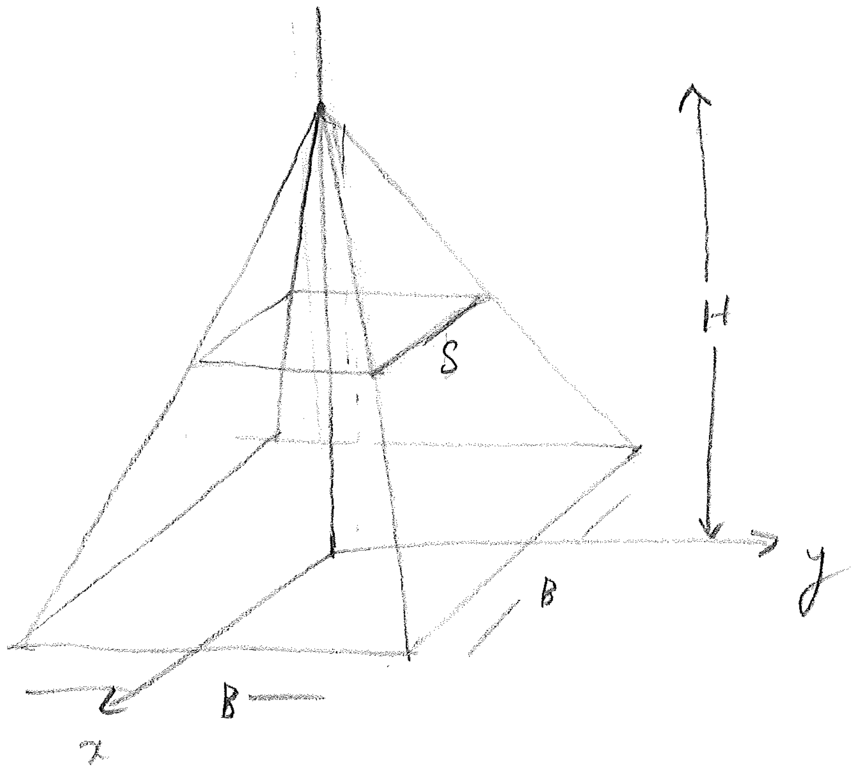
Add the slices together to get an approx. for the total volume V

$$V \approx \sum_{i=1}^{n} A(h_{i-1})\,\Delta h.$$

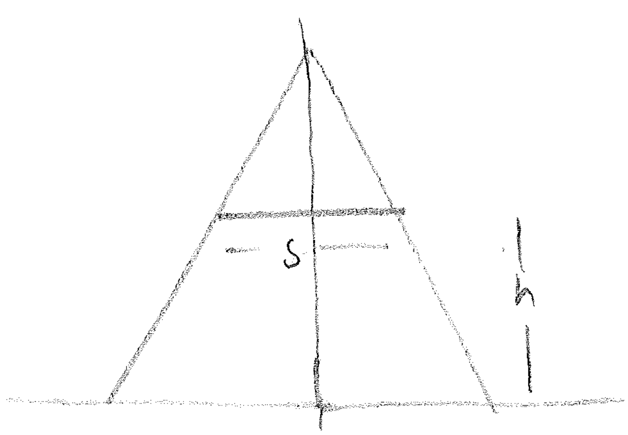
In the limit this Riemann sum becomes an integral and we define V by

$$V = \int_a^b A(h)\,dh$$

Ex. Pyramid with square base of side length B and height H.



Let $s = s(h)$ be the side length of the square cross-section at height h.



$s(h)$ is a linear fn of h where

$s(0) = B$ and $s(h) = 0$.

Implies that $s(h) = B - \frac{B}{H} \cdot h$

$$= B(1 - h/H)$$

Then $A(h) = s^2 = B^2(1 - h/H)^2$

$$= B^2\left(1 - \frac{2h}{H} + h^2/H^2\right)$$

and

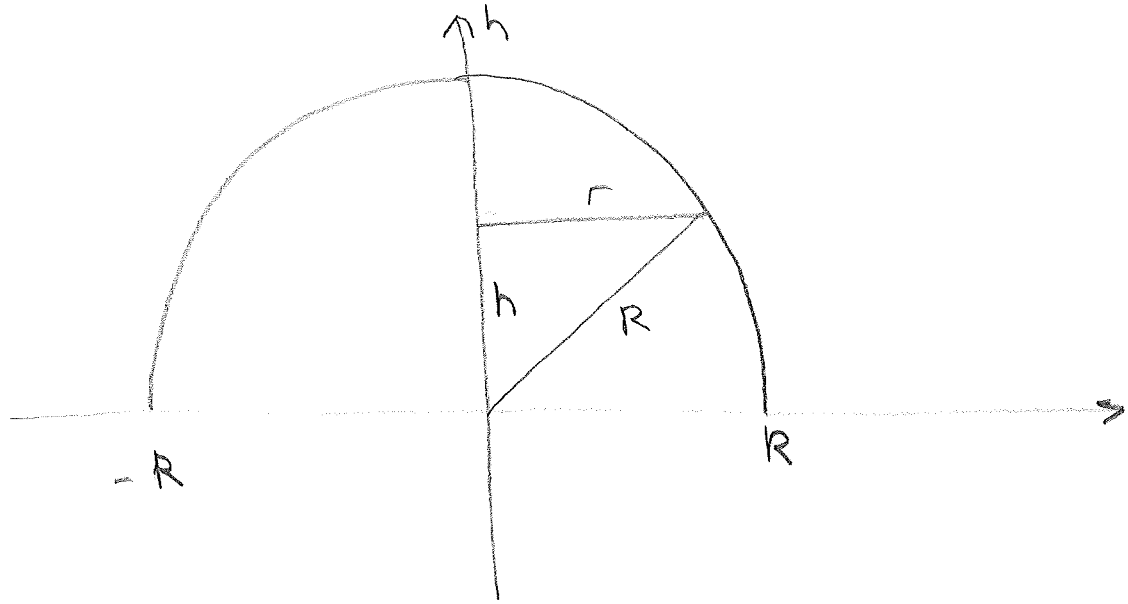
$$V = \int_0^H B^2\left(1 - \frac{2h}{H} + \frac{h^2}{H^2}\right)dh$$

$$= B^2 \int_0^H \left(1 - \frac{2h}{H} + \frac{h^2}{H^2}\right)dh$$

$$= B^2 \left[h - \frac{h^2}{H} + \frac{h^3}{3H^2} \right]_0^H$$

$$= B^2\left(\left(H - \frac{H^2}{H} + \frac{H^3}{3H^2}\right) - 0\right) = \frac{B^2 H}{3}.$$

Ex. Hemisphere of radius R



Similarly to before

$$r = \sqrt{R^2 - h^2}$$

$$A(h) = \pi r^2 = \pi (R^2 - h^2)$$

$$V = \int_0^R A(h)\,dh = \pi \int_0^R (R^2 - h^2)\,dh$$

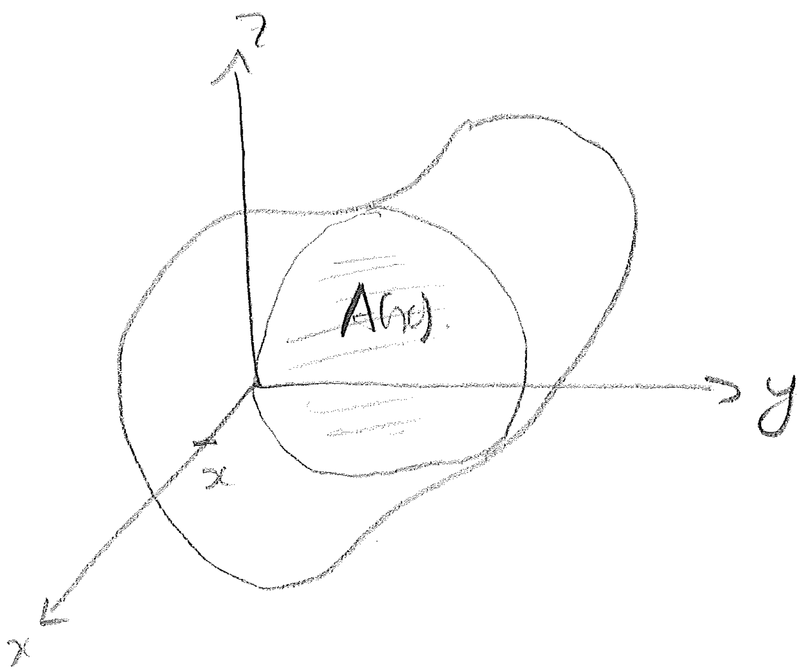
$$= \pi \left[R^2 h - \frac{h^3}{3} \right]_0^R$$

$$= \pi \left(\left(R^3 - \frac{R^3}{3}\right) - 0 \right)$$

$$= \underline{\frac{2\pi R^3}{3}}$$ as we'd expect.

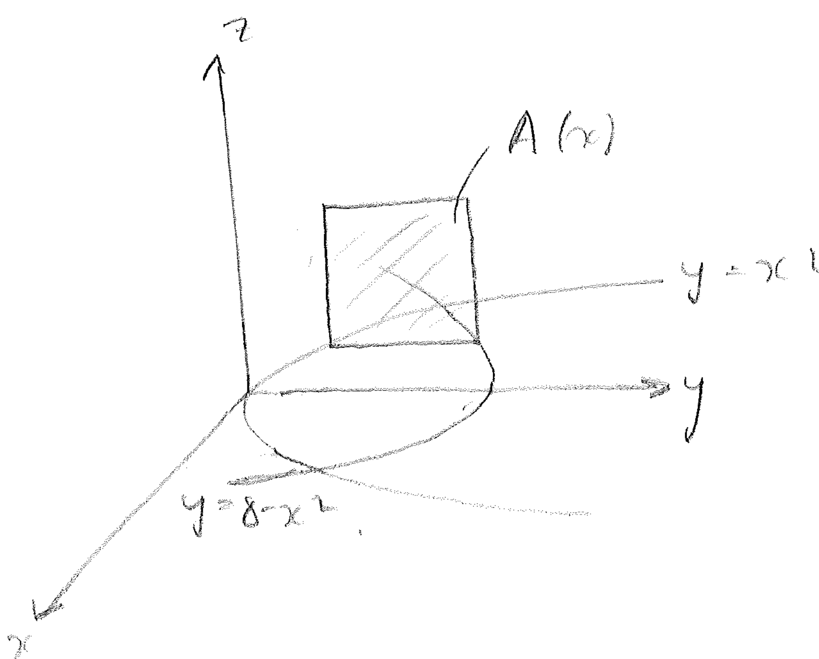
Volume by Vertical Cross-Sections

If we slice the region by vertical slices perpendicular to the x-axis whose area is $A(x)$, we get a similar formula



$$V = \int_a^b A(x)\,dx.$$

Ex. Find the volume of the solid whose base is the region in the x-y plane bounded by the parabolas $y = x^2$ and $y = 8 - x^2$ and whose cross-sections are squares perpendicular to the x-axis with one side in the x-y plane



As in previous example, x runs from -2 to 2.

Also, the square has side length

$$s = 8 - 2x^2.$$

Then $A(x) = s^2 = (8-2x^2)^2$

$$= 64 - 32x^2 + 4x^4.$$

$$V = \int_{-2}^{2} (64 - 32x^2 + 4x^4)\,dx$$

$$= \left[64x - \frac{32}{3}x^3 + \frac{4}{5}x^5 \right]_{-2}^{2}$$

$$= 64(2) - \frac{32(8)}{3} + \frac{4}{5}(32)$$

$$- \left(64(-2) - \frac{32(-8)}{3} + \frac{4}{5}(-32) \right)$$

$$= 128 - \frac{256}{3} + \frac{128}{5}$$

$$+128 - \frac{256}{3} + \frac{128}{5}$$

$$= \frac{2048}{15} \approx 136.5.$$