§ 7.8 Comparison of Improper Integrals

Often we can tell whether or not an improper integral converges, even when we can't find an antiderivative.

This is done by comparing the improper integral with that of a simpler $f^{\underline{n}}$.

<u>Ex</u>. $\displaystyle\int_1^\infty \frac{1}{\sqrt{x^3+5}}\,dx.$

When x is large $x^3 \gg 5$ and so

$$\frac{1}{\sqrt{x^3+5}} \approx \frac{1}{\sqrt{x^3}} = \frac{1}{x^{3/2}}.$$

Suggests we compare with $\frac{1}{x^{3/2}}$.

Now $\displaystyle\int_1^\infty \frac{1}{x^{3/2}}\,dx$ converges $(p = 3/2 > 1)$

and has value $\dfrac{1}{3/2 - 1} = 2$ (see last section).

Also, for $x \geq 1$

$$0 \leq \frac{1}{\sqrt{x^3+5}} \leq \frac{1}{x^{3/2}}.$$

Thus, we can conclude that for any $b > 1$,

$$0 \leq \int_1^b \frac{1}{\sqrt{x^3+5}}\,dx \leq \int_1^b \frac{1}{x^{3/2}}\,dx.$$

It then follows that

$$\int_1^b \frac{1}{\sqrt{x^3+5}}\,dx$$

must have a finite limit as $b \to \infty$.

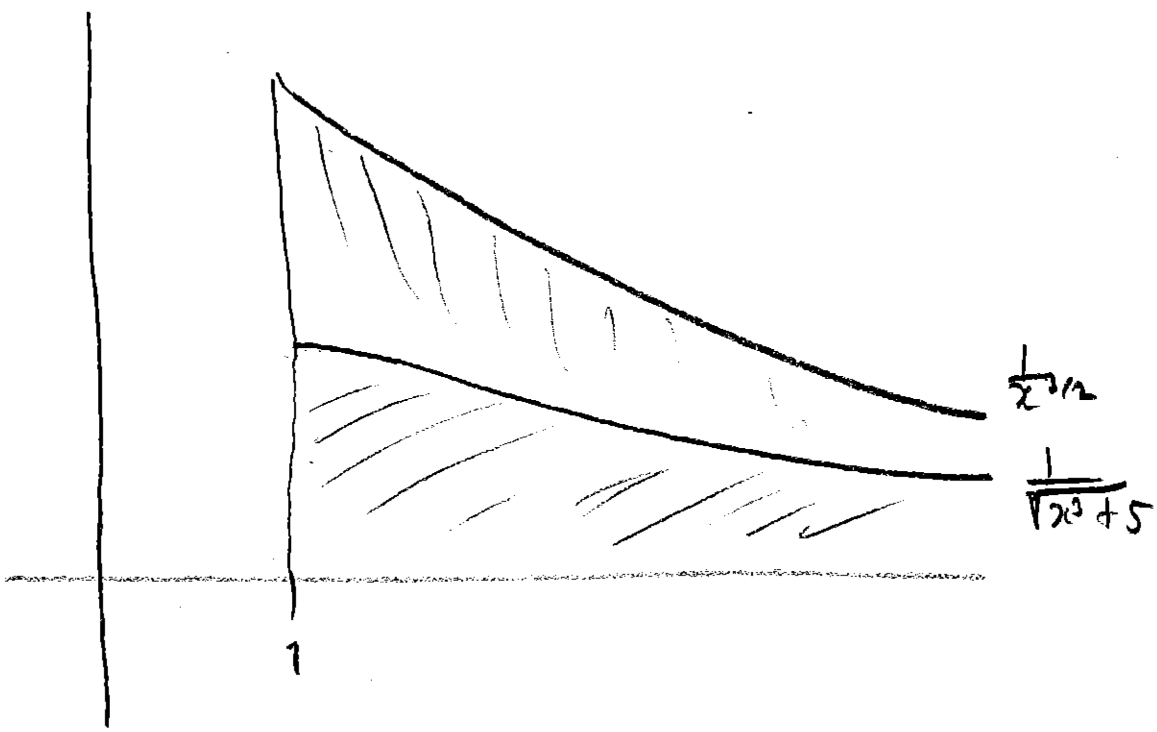
Thus

$$\int_1^\infty \frac{1}{\sqrt{x^3+5}}\,dx$$

converges and in fact we can say that

$$0 \leqslant \int_1^\infty \frac{1}{\sqrt{x^3+5}}\,dx \leqslant 2.$$

In terms of areas



Ex. Estimate the value of

$$\int_1^\infty \frac{1}{\sqrt{x^3+5}}\,dx$$

with an error of < 0.01.

Write

$$\int_1^\infty \frac{1}{\sqrt{x^3+5}}\,dx = \int_1^b \frac{1}{\sqrt{x^3+5}}\,dx + \int_b^\infty \frac{1}{\sqrt{x^3+5}}\,dx$$

Idea is to choose b large enough so that the 'tail' of the integral $\int_b^\infty \frac{1}{\sqrt{x^3+5}}\,dx$ satisfies

$$\left| \int_b^\infty \frac{1}{\sqrt{x^3+5}}\,dx \right| < 0.01$$

Now

$$0 < \int_b^\infty \frac{1}{\sqrt{x^3+5}}\,dx < \int_b^\infty \frac{1}{x^{3/2}} = \frac{2}{\sqrt{b}}.$$

So we'll have what we want provided

$$\frac{2}{\sqrt{b}} < .01$$

$$\frac{\sqrt{b}}{2} > 100$$

$$\sqrt{b} > 200$$

$$b > 40{,}000.$$

Thus with an error of < 0.01,

$$\int_1^\infty \frac{1}{\sqrt{x^3+5}}\,dx \approx \int_1^{40{,}001} \frac{1}{\sqrt{x^3+5}}\,dx = 1.669.$$

The Comparison Test for $\int_a^\infty f(x)\,dx$

Let $f(x) \geqslant 0$. Comparison involves two steps.

1. Guess, by looking at the integrand for large x, what the integrand 'behaves' like and whether it converges or not. This gives us a fn to compare the integrand with.

2. Confirm the guess by comparison.

- If $0 \leq f(x) \leq g(x)$ and $\int_a^\infty g(x)$ converges, then $\int_a^\infty f(x)\,dx$ converges.

- If $0 \leq g(x) \leq f(x)$ and $\int_a^\infty g(x)$ diverges, then $\int_a^\infty f(x)\,dx$ diverges.

Similar comparisons exist for other types of improper integral.

<u>Ex.</u> $$\int_4^\infty \frac{dt}{\ln t - 1}$$

When t is very large $\ln t >> 1$ and so

$$\frac{1}{\ln t - 1} \approx \frac{1}{\ln t}$$

Probably diverges as $\ln t$ only grows slowly.
So we try to make $\frac{1}{\ln t - 1}$ bigger than a $f^{\underline{n}}$ $g(t)$ where $\int_4^\infty g(t)\,dt$ diverges.

Now, for $t \geqslant 4$,

$$\frac{1}{\ln t - 1} > \frac{1}{\ln t} > \frac{1}{t} > 0 \quad \left[\begin{array}{l} t > \ln t \text{ for} \\ t > 0 \end{array} \right]$$

and since $\int_4^\infty \frac{1}{t}\,dt$ diverges,

$\int_4^\infty \frac{dt}{\ln t - 1}$ also diverges by the comparison test.

Useful Integrals to Compare With

$$\int_1^\infty \frac{1}{x^p}\,dx \quad \begin{cases} \text{converges for } p>1 \\ \text{diverges for } p\leq 1 \end{cases}$$

$$\int_0^1 \frac{1}{x^p}\,dx \quad \begin{cases} \text{converges for } p<1 \\ \text{diverges for } p\geq 1. \end{cases}$$

$$\int_0^\infty e^{-ax}\,dx \quad \begin{cases} \text{converges for } a>0 \\ \text{diverges for } a<0 \end{cases}$$

Ex. $\displaystyle\int_1^\infty \frac{\sin x + 3}{\sqrt{x}}\,dx$.

Since $-1 \le \sin x \le 1$, -- for $x \ge 1$

$$0 \le \frac{2}{\sqrt{x}} \le \frac{\sin x + 3}{\sqrt{x}} \le \frac{4}{\sqrt{x}}.$$

Since $\displaystyle\int_1^\infty \frac{2}{\sqrt{x}}\,dx$ diverges,

$\displaystyle\int_1^\infty \frac{\sin x + 3}{\sqrt{x}}\,dx$ also diverges by the comparison test.

<u>Note</u> The fact that $\dfrac{\sin x + 3}{\sqrt{x}} \le \dfrac{4}{\sqrt{x}}$ is of no use!

Ex. $$\int_1^\infty e^{-x^2/2}\,dx$$

Integrand decreases very rapidly in x

$$\Downarrow$$

Expect convergence.

Thus we should compare the integrand with a larger fn whose integral converges.

For example, might try $\int_1^\infty e^{-x}\,dx$ which does converge.

For $x \geq 2$,

$$\frac{x^2}{2} \geq x$$

$$-\frac{x^2}{2} \leq -x$$

so $$0 < e^{-x^2/2} < e^{-x}.$$

This would allow us to say that

$$\int_2^\infty e^{-x^2/2}\,dx$$ converges, but we

wanted $$\int_1^\infty e^{-x^2}\,dx$$ and the comparison

doesn't work for $1 \leq x < 2$.

Get round this problem by splitting up the integral

$$\int_1^\infty e^{-x^2/2}\,dx = \int_1^2 e^{-x^2/2}\,dx + \int_2^\infty e^{-x^2/2}\,dx$$

First integral is simply an ordinary definite integral of a finite cts fn on a closed interval, so no problems while we already know that the second integral converges.

Thus $$\int_1^\infty e^{-x^2/2}\,dx$$ converges.

A useful shortcut for testing convergence of improper integrals.

<u>Limit Comparison Test for Integrals (Type I)</u>

If $f(x)$, $g(x)$ are continuous and positive on $[a, \infty)$ and

$$\lim_{x \to \infty} \frac{f(x)}{g(x)} = C \quad \text{exists with } 0 < C < \infty,$$

then

$$\int_a^\infty f(x)\,dx \quad \text{and} \quad \int_a^\infty g(x)\,dx$$

either both converge or both diverge.

Similar versions exist for integrals of the type $\int_{-\infty}^{b} f(x)\,dx$ and integrals where the integrand becomes unbounded (Type II, III).

Ex. $\displaystyle\int_2^\infty \frac{1}{x^2-1}\,dx.$

Natural integral to compare with is

$$\int_2^\infty \frac{1}{x^2}\,dx.$$

$$\lim_{x\to\infty} \frac{\frac{1}{x^2-1}}{\frac{1}{x^2}} = \lim_{x\to\infty} \frac{x^2}{x^2-1}$$

$$= \lim_{x\to\infty} \frac{1}{1-\frac{1}{x^2}}$$

$$= 1 > 0.$$

Since $\displaystyle\int_2^\infty \frac{1}{x^2}\,dx$ converges,

$\displaystyle\int_2^\infty \frac{1}{x^2-1}\,dx$ also converges by the limit comparison test.

N.b. This is easier to use here than the regular comparison test as

$$\frac{1}{x^2-1} \geq \frac{1}{x^2} \quad , \quad x \geq 2.$$

which goes the wrong way for us to be able to use.