§ 7.4 Algebraic Identities and Trigonometric Substitutions

Part I – The method of partial fractions

This is a method for rewriting certain rational fns in a way which makes them easier to integrate.

Ex. Find A, B for which

$$\frac{1}{(x-2)(x-5)} = \frac{A}{x-2} + \frac{B}{x-5}$$

Put the rhs. over a common denominator

$$= \frac{A(x-5)}{(x-2)(x-5)} + \frac{B(x-2)}{(x-2)(x-5)}.$$

So

$$\frac{1}{(x-2)(x-5)} = \frac{A(x-5) + B(x-2)}{(x-2)(x-5)}$$

Multiply both sides by $(x-2)(x-5)$

$$1 = A(x-5) + B(x-2)$$

$$1 = (A+B)x - 5A - 2B$$

Lhs & rhs are both polys. in x and they are equal iff the coefficients of the different powers of x match.

So

x/ $\quad A + B = 0 \quad$ ①

1/ $\quad -5A - 2B = 1 \quad$ ②

① $\Rightarrow$ $B = -A$ & if we subst in ②

we get

$$-5A + 2A = 1$$

$$-3A = 1$$

$$A = -\frac{1}{3}$$

$$B = -A = \frac{1}{3}.$$

So $$\frac{1}{(x-2)(x-5)} = -\frac{1}{3(x-2)} + \frac{1}{3(x-5)}$$

Now let's see why this is so useful....

Ex. $$\int \frac{1}{(x-2)(x-5)}\,dx$$

From above we can rewrite this as

$$\int \left(\frac{-1}{3(x-2)} + \frac{1}{3(x-5)} \right) dx$$

$$= -\frac{1}{3}\int \frac{dx}{x-2} + \frac{1}{3}\int \frac{dx}{x-5}$$

$$= -\frac{1}{3}\ln|x-2| + \frac{1}{3}\ln|x-5| + C$$ by V-26 from table.

Ex $\int \frac{x+2}{x^2+x}\,dx$

Factor the denominator x^2+x as $x(x+1)$ and write

$$\frac{x+2}{x^2+x} = \frac{x+2}{x(x+1)} = \frac{A}{x} + \frac{B}{x+1}$$

$$= \frac{A(x+1) + Bx}{x(x+1)}$$

So $x+2 = A(x+1) + Bx$

$= (A+B)x + A$

Thus

$x/$ $\quad A+B = 1 \quad (1)$

$1/$ $\quad A = 2 \quad (2)$

Sub for A in (1)

$2 + B = 1$

$B = -1$

Thus

$$\frac{x+2}{x^2+x} = \frac{2}{x} - \frac{1}{x+1}$$

and so

$$\int \frac{x+2}{x^2+x}\,dx = 2\int \frac{dx}{x} - \int \frac{dx}{x+1}$$

$$= 2\ln|x| - \ln|x+1| + C$$

Ex

$$\int \frac{10x - 2x^2}{(x-1)^2(x+3)}$$

NOTE Although the $(x-1)$ term is squared, we will need a $\frac{1}{x-1}$ term in our partial fraction expansion as well as a $\frac{1}{(x-1)^2}$ term

So write

$$\frac{10x - 2x^2}{(x-1)^2(x+3)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+3}$$

$$= \frac{A(x-1)(x+3)}{(x-1)^2(x+3)} + \frac{B(x+3)}{(x-1)^2(x+3)} + \frac{C(x-1)^2}{(x-1)^2(x+3)}$$

$$= \frac{A(x^2+2x-3) + B(x+3) + C(x^2-2x+1)}{(x-1)^2(x+3)}$$

$$= \frac{(A+C)x^2 + (2A+B-2C)x + (-3A+3B+C)}{(x-1)^2(x+3)}$$

Compare powers of x.

$$x^2: \quad A + C = -2 \quad (1)$$

$$x: \quad 2A + B - 2C = 10 \quad (2)$$

$$1: \quad -3A + 3B + C = 0 \quad (3)$$

Solve by Gaussian elimination

Take 2(1) from (2) and add 3(1) to (3) to get rid of A in (2) & (3)

$$A + C = -2 \quad (1)$$

$$B - 4C = 14 \quad (2)'$$

$$3B + 4C = -6 \quad (3)'$$

Take 3(2)' from (3)' to get rid of B in (3)'

$$A + C = -2 \quad (1)$$

$$B - 4C = 14 \quad (2)'$$

$$16C = -48 \quad (3)''$$

Now do back substitution to get C, B, A in turn.

From ③'', $C = -3$

sub for C into ②'

$$B - 4(-3) = 14$$

$$B = 2$$

Sub for ~~B~~, C in ①

$$A - 3 = -2$$

$$A = 1.$$

So $A = 1$, $B = 2$, $C = -3$ and

$$\frac{10x - 2x^2}{(x-1)^2(x+3)} = \frac{1}{x-1} + \frac{2}{(x-1)^2} - \frac{3}{x+3}$$

The actual integration is now fairly easy

$$\int \frac{10x - 2x^2}{(x-1)^2(x+3)}\,dx = \int \left(\frac{1}{x-1} + \frac{2}{(x-1)^2} - \frac{3}{x+3} \right) dx$$

$$= \int \frac{dx}{x-1} + 2\int \frac{dx}{(x-1)^2} - 3\int \frac{dx}{x+3}$$

In the first two integrals let

$w = x-1, \quad dw = dx$

and in the third let

$y = x+3, \quad dy = dx$

Get.

$$\int \frac{dw}{w} + 2\int \frac{dw}{w^2} - 3\int \frac{dy}{y}$$

$$= \ln|w| + 2\,\frac{w^{-1}}{(-1)} - 3\ln|y| + C$$

Convert back to x and tidy

$$= \ln|x-1| - \frac{2}{(x-1)} - 3\ln|x+3| + C.$$

Ex. $\displaystyle\int \frac{2x^2 - x - 1}{(x^2+1)(x-2)}\,dx$

Here x^2+1 cannot be factored any further (using only real numbers). The correct partial fractions expansion to look for in this case is of the form

$$\frac{2x^2 - x - 1}{(x^2+1)(x-2)} = \frac{Ax+B}{x^2+1} + \frac{C}{x-2}$$

$$= \frac{(Ax+B)(x-2) + C(x^2+1)}{(x^2+1)(x-2)}$$

$$= \frac{Ax^2 + (-2A+B)x - 2B + Cx^2 + C}{(x^2+1)(x-2)}$$

$$= \frac{(A+C)x^2 + (-2A+B)x - 2B + C}{(x^2+1)(x-2)}$$

Equating powers of x gives

x^2/

$$\begin{aligned} A \qquad\quad &+ C = 2 \qquad (1) \\ -2A + B \qquad &= -1 \qquad (2) \\ -2B &+ C = -1 \qquad (3) \end{aligned}$$

Gaussian elimination (again!)

Add 2(1) to (2) to get rid of A in (2), (3)

$$\begin{aligned} A \qquad\quad &+ C = 2 \qquad (1) \\ B &+ 2C = 3 \qquad (2)' \\ -2B &+ C = -1 \qquad (3)' \end{aligned}$$

Add 2(2)' to (3)' to get rid of B in (3)'

$$\begin{aligned} A \qquad\quad &+ C = 2 \qquad (1) \\ B &+ 2C = 3 \qquad (2)' \\ &5C = 5 \qquad (3)'' \end{aligned}$$

Back substitution

(3)'' $\Rightarrow$ $C = 1$

Sub for C in (2)'

$$B + 2 = 3$$

$$B = 1$$

Sub for B, C in (1)

$$A + 1 = 2$$

$$A = 1.$$

So $A = B = C = 1$ and

$$\frac{2x^2 - x - 1}{(x^2+1)(x-2)} = \frac{x+1}{x^2+1} + \frac{1}{x-2}.$$

Thus

$$\int \frac{2x^2 - x - 1}{(x^2+1)(x-2)}\,dx = \int \frac{x+1}{x^2+1}\,dx + \int \frac{dx}{x-2}$$

Split up the first integral further to get

$$= \int \frac{x\,dx}{x^2+1} + \int \frac{1 \cdot dx}{x^2+1} + \int \frac{dx}{x-2}$$

In the first integral let $w = x^2+1$,

so

$$dw = 2x\,dx \quad \text{and} \quad \frac{dw}{2} = x\,dx$$

Get

$$\frac{1}{2}\int \frac{dw}{w} + \int \frac{dx}{x^2+1} + \int \frac{dx}{x-2}$$

$$= \frac{1}{2}\ln|w| + \frac{1}{1}\arctan\left(\frac{x}{1}\right) + \ln|x-2| + C$$

Convert back to x

$$= \frac{1}{2}\ln|x^2+1| + \arctan x + \ln|x-2| + C.$$

Ex.

$$\int \frac{x^3 - 7x^2 + 10x + 1}{x^2 - 7x + 10}\, dx$$

Here the degree of the numerator is $\geq$ that of the denominator.

As in the last section, the first thing to do here is algebraic long division

$$\begin{array}{r|l} & x \\ \hline x^2 - 7x + 10 & x^3 - 7x^2 + 10x + 1 \\ & \underline{x^3 - 7x^2 + 10x} \\ & \qquad\qquad\qquad\quad 1. \end{array}$$

So

$$\frac{x^3 - 7x^2 + 10x + 1}{x^2 - 7x + 10} = \frac{x(x^2 - 7x^2 + 10) + 1}{x^2 - 7x + 10}$$

$$= x + \frac{1}{x^2 - 7x + 10}$$

If we factor $x^2 - 7x + 10$ as $(x-2)(x-5)$, we get

$$\frac{x^3 - 7x^2 + 10x + 1}{x^2 - 7x + 10} = x + \frac{1}{(x-2)(x-5)}$$

We then use partial fractions on $\frac{1}{(x-2)(x-5)}$.

From earlier

$$\frac{1}{(x-2)(x-5)} = -\frac{1}{3(x-2)} + \frac{1}{3(x-5)}$$

and so

$$\frac{x^3 - 7x^2 + 10x + 1}{x^2 - 7x + 10} = x - \frac{1}{3(x-2)} + \frac{1}{3(x-5)}$$

Thus

$$\int \frac{x^3 - 7x^2 + 10x + 1}{x^2 - 7x + 10}\, dx$$

$$= \int x\, dx - \frac{1}{3}\int \frac{dx}{x-2} + \frac{1}{3}\int \frac{dx}{x-5}$$

$$= \frac{x^2}{2} - \frac{1}{3}\ln|x-2| + \frac{1}{3}\ln|x-5| + C$$

Partial fractions works for many, but not all rational functions.

Strategy for Integrating a Rational Function $\frac{P(x)}{Q(x)}$.

1. If $\deg P \geq \deg Q$, use algebraic long division and use partial fractions on the remainder.

2. If $Q(x)$ is a product of distinct linear factors, use partial fractions of the form

$$\frac{A}{x-c}$$

3. If $Q(x)$ has a repeated linear factor $(x-c)^n$, use partial fractions of the form

$$\frac{A_1}{x-c} + \frac{A_2}{(x-c)^2} + \cdots + \frac{A_{n-1}}{(x-c)^n}$$

(the same n as the power of the factor itself)

4. If $Q(x)$ contains an unfactorable quadratic factor $q(x)$, try a partial fraction of the form

$$\frac{Ax+B}{q(x)}.$$

We will see more about how to integrate factors of the type

$$\frac{Ax+B}{q(x)}$$

soon.