§ 8.4 Density and Centre of Mass

Density measures the amount of some quantity per unit size

e.g. kg/m^3, lb/ft, people/square mile

To find the total amount of some quantity from the density

i) Divide the region into small pieces on each of which the density is approx. constant.

The amount for each piece is approx. the density (somewhere in the piece) multiplied by the size of the piece

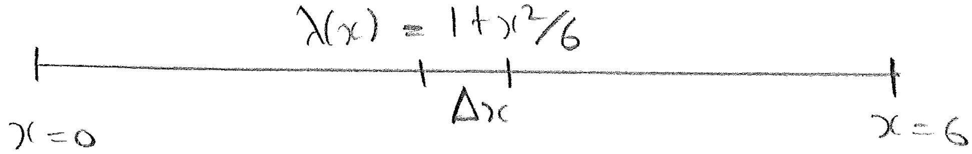
ii) Add up all the contributions for all the pieces.

This gives us a Riemann sum which approximates the total amount.

iii) Let the number of pieces tend to infinity in a suitable way.

This gives us an integral which we <u>define</u> to be the total amount.

Ex A 6ft long rod lies on the x-axis with its left end at $x = 0$. If the linear density of the rod is $\lambda(x) = 1 + \frac{x^2}{6}$ lbs/ft, find the mass of the rod.



The mass of a small piece is

$$\lambda(x)\Delta x = (1 + x^2/6)\Delta x$$

If we add together the masses of the small pieces, we get

$$M \approx \sum_{i=0}^{n-1} \lambda(x_{i-1})\Delta x = \sum_{i=0}^{n-1} (1 + x^2/6)\Delta x$$

Letting $n \to \infty$, we get

$$M = \int_0^6 \lambda(x)\,dx = \int_0^6 (1 + x^2/6)\,dx$$

$$= \left[x + \frac{x^3}{18} \right]_0^6$$

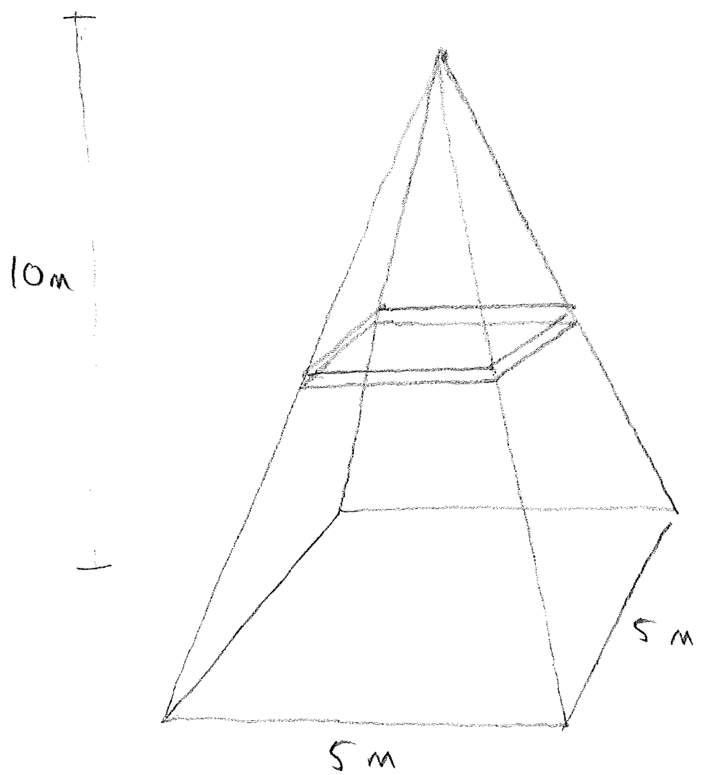
$$= 6 + \frac{216}{18} - 0$$

$$= 6 + 12$$

$$= 18 \text{ lbs.}$$

<u>NOTE</u> Don't forget to give the correct units at the end!

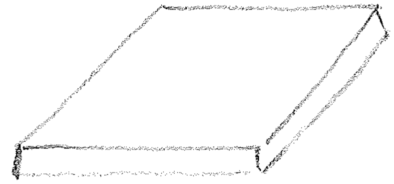
Ex A pyramid is 10m tall and has a square base of side length 5m. If the pyramid is made of a stone whose density is $p(h) = 10 - \frac{h}{4}$ tonnes/m where h is the height in metres above the base, find the mass of the pyramid.



Slice the pyramid horizontally (why?)

The slice is approx a square slab of thickness Δh and side length

$$s = 5(1 - h/10) = 5 - h/2 \text{ m.}$$



The mass of this slice is then approx.

$$s^2 \Delta h \cdot p(h) = (5 - h/2)^2 \cdot \Delta h \cdot (10 - h/4).$$

$$= (5 - \tfrac{h}{2})^2 \, (10 - \tfrac{h}{4}) \, \Delta h$$

$$= (5 - \tfrac{h}{2})^2 \cdot \tfrac{1}{2} \, (20 - \tfrac{h}{2}) \, \Delta h$$

$$= \tfrac{1}{2} \left((5 - h/2)^2 \, (15 + (5 - h/2) \right) \Delta h$$

Get the integral

$$M = \frac{1}{2} \int_0^{10} ((5 - h/2)^2 (15 + (5 - h/2)) \, dh$$

Let $w = 5 - h/2$ $\quad dw = -\frac{1}{2} dh$

When $h = 0$, $w = 5$ $\quad h = 10, w = 0$

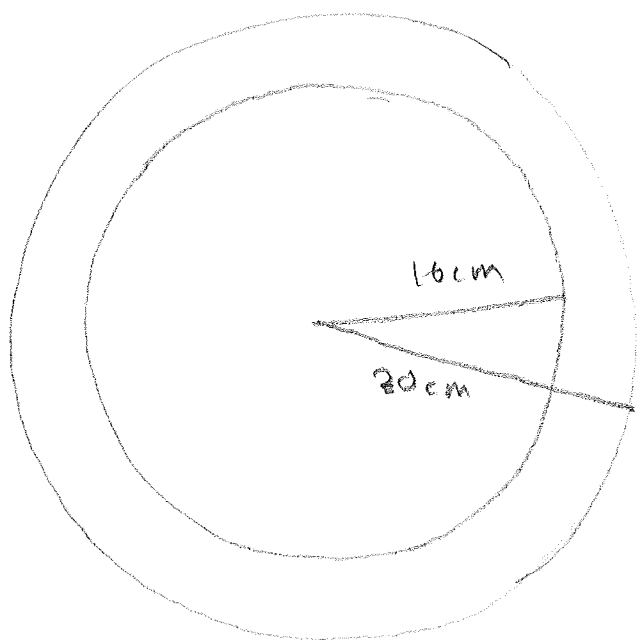
$$= \frac{1}{2} \int_5^0 w^2 (15 - w) \cdot - \frac{dw}{2}$$

$$= \frac{1}{4} \int_0^5 (15w^2 - w^3) \, dw$$

$$= \frac{1}{4} \left[5w^3 - \frac{w^4}{4} \right]_0^5 = \frac{1}{4} \left(625 - \frac{625}{4} - 0 \right)$$

$$\approx 117 \text{ tonnes.}$$

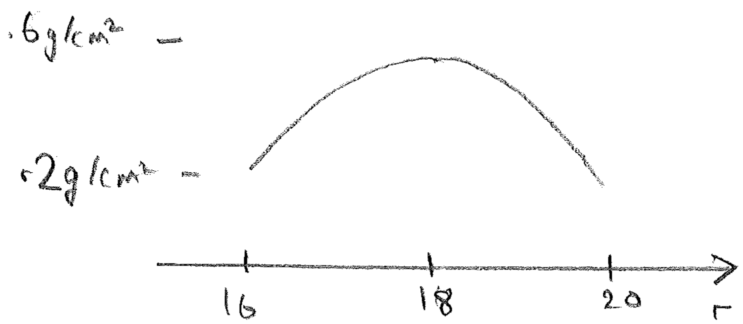
Ex A flying ring is in the form of a thin annulus of inner radius 16 cm and outer radius 20 cm. If the areal density of the ring as a function of the radius is $\sigma(r) = \frac{6-(r-18)^2}{10}$ grams/cm^2. Find the mass of the ring.



Here a small piece is a very thin ring of radius r and with Δr.

The area of this piece is approx the circumference $\times$ Δr ie

$$2\pi r \Delta r$$



Density Profile

and the mass is approx

$$\Delta M = 2\pi r \sigma(r) \Delta r.$$

$$= 2\pi r \left(\frac{6-(r-18)^2}{10}\right) \Delta r.$$

$$= \frac{2\pi r}{10}\left(6 - (r^2 - 36r + 324)\right)\Delta r$$

$$= \frac{2\pi r}{10}\left(6 - r^2 + 36r - 320\right)\Delta r$$

$$= \frac{2\pi r}{10}\left(-r^2 + 36r - 318\right)\Delta r$$

$$= \frac{\pi}{5}\left(-r^3 + 36r^2 - 318r\right)\Delta r$$

The total mass is then approx.

$$M \approx \sum_{i=0}^{n-1} \frac{\pi}{5}\left(-r_{i-1}^3 + 36r_{i-1}^2 - 318\, r_{i-1}\right)\Delta r$$

and in the limit

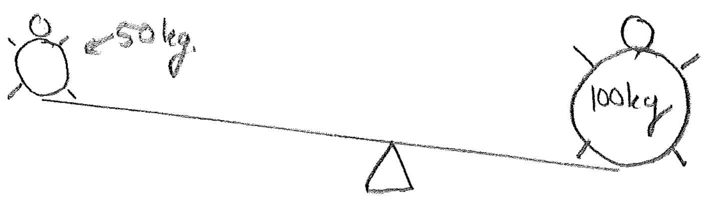
$$M = \frac{\pi}{5}\int_{16}^{20} \left(-r^3 + 36r^2 - 318\, r\right)dr$$

$$= \frac{\pi}{5}\left[\frac{-r^4}{4} + 12r^3 - 159\, r^2\right]_{16}^{20}$$

$$= 67.2\pi \text{ grams} \approx 211 \text{ grams.}$$

Centre of Mass

A child of 50kg and a child of 100kg sit on a seesaw



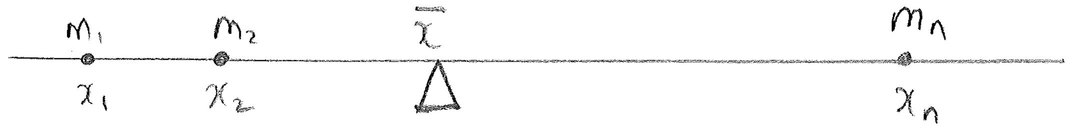
In order to balance, the heavy child needs to be twice as close to the fulcrum as the light child, e.g. if the light child is 2m away from the fulcrum, then the heavy child must be 1 m away.

What is the same for both children is the product of the mass and the distance from the fulcrum, ie the moments about the fulcrum

$$50kg \times 2m = 100kg \times 1m.$$

Suppose now we have n masses $m_1, m_2, \ldots, m_n$ in a line at positions $x_1, x_2, \ldots, x_n$ (resp.) on the x-axis.

We want to find where to put our fulcrum $\bar{x}$ so that the whole system balances



For a given mass m_i, the moment of this mass about $\bar{x}$ is

$$m_i(x_i - \bar{x})$$

n.b. this is positive if x_i is to the right of $\bar{x}$ and negative if x_i is to the left.

For the system to balance, the sum of all the moments about $\bar{x}$ must be zero.

ie $$\sum_{i=1}^{n} m_i(x_i - \bar{x}) = 0$$

So

$$\sum_{i=1}^{n} m_i x_i - \sum_{i=1}^{n} m_i \bar{x} = 0$$

$$\sum_{i=1}^{n} m_i x_i - \bar{x} \sum_{i=1}^{n} m_i = 0$$

$$\bar{x} \sum_{i=1}^{n} m_i = \sum_{i=1}^{n} m_i x_i$$

So $$\boxed{\bar{x} = \frac{\sum_{i=1}^{n} m_i x_i}{\sum_{i=1}^{n} m_i} = \frac{\sum_{i=1}^{n} m_i x_i}{M}}$$

Where $M = \sum_{i=1}^{n} m_i$ is the total mass.

Can think of $\bar{x}$ as the weighted average position of the whole system

Ex. The two children again.

Set $m_1 = 50\,\text{kg}, \quad x_1 = 0$

$m_2 = 100\,\text{kg}, \quad x_2 = 3.$

$$\bar{x} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}$$

$$= \frac{50(0) + 100(3)}{50 + 100}$$

$$= \frac{300}{150} = 2\,\text{m}$$ as we'd expect.

For objects which cover entire regions and not just a number of points, these sums become integrals.

e.g. For an object of linear density $\lambda(x)$ which lies between $x=a$ and $x=b$, we have

$$\bar{x} = \frac{\int_a^b x\,\lambda(x)\,dx}{\int_a^b \lambda(x)\,dx} = \frac{\int_a^b x\,\lambda(x)\,dx}{M}.$$

<u>Ex</u> For a uniform rod of constant density λ which extends from $x=a$ to $x=b$, the centre of mass is found by getting

$$\int_a^b x\,\lambda(x)\,dx = \int_a^b x\,\lambda\,dx = \left[\frac{\lambda x^2}{2}\right]_a^b = \frac{\lambda}{2}(b^2-a^2)$$

$$\int_a^b \lambda(x)\,dx = \int_a^b \lambda\,dx = \left[\lambda x\right]_a^b = \lambda(b-a).$$

So $$\bar{x} = \frac{\int_a^b x\,\lambda(x)\,dx}{\int_a^b \lambda(x)\,dx}$$

$$= \frac{\frac{\lambda}{2}(b^2-a^2)}{\lambda(b-a)}$$

$$= \frac{1}{2}\,\frac{(b-a)(b+a)}{b-a}$$

$$= \frac{b+a}{2}$$ as we'd expect.

Ex. The 6ft rod of linear density $\lambda(x) = 1 + \frac{x^2}{6}$ lbs / ft of last time.

Already found that $M = 18$ lbs.

Now $\int_a^b x\,\lambda(x)\,dx$

$$= \int_0^6 x\left(1 + \frac{x^2}{6}\right) dx$$

$$= \int_0^6 \left(x + \frac{x^3}{6}\right) dx$$

$$= \left[\frac{x^2}{2} + \frac{x^4}{24}\right]_0^6$$

$$= \frac{6^2}{2} + \frac{6^4}{24} - 0$$

$$= 18 + 54 = 72 \text{ ft lbs.}$$

Thus $\bar{x} = \dfrac{\int_a^b x\,\lambda(x)\,dx}{M} = \dfrac{72}{18}\dfrac{\text{ft lbs}}{\text{lbs}} = 4 \text{ ft.}$

For a 2D region lying between $x = a$ and $x = b$ and $y = c$ and $y = d$, the coords of the centre of mass are given by

$$\bar{x} = \frac{\int_a^b x\delta(x)\,dx}{\int_a^b \delta(x)\,dx} = \frac{\int_a^b x\delta(x)}{M}$$

$$\bar{y} = \frac{\int_c^d y\delta(y)\,dy}{\int_c^d \delta(y)\,dy} = \frac{\int_c^d y\delta(y)\,dy}{M}.$$

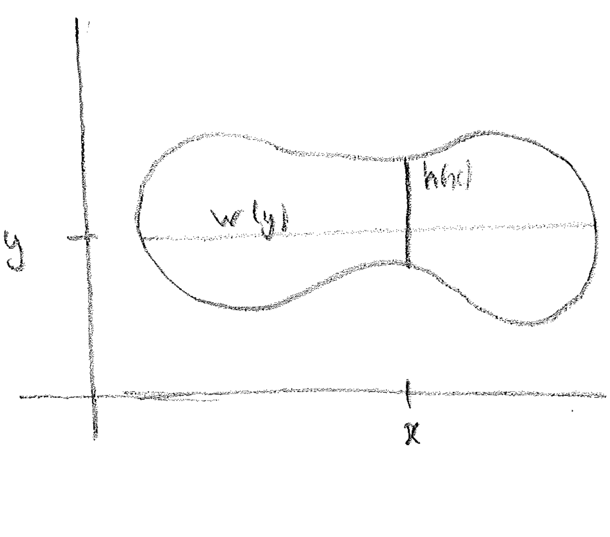
Here $\delta(x)$, $\delta(y)$ refer to the mass per unit length in the x- and y-directions respectively.

An important special case is when we have a uniform region with constant density σ

$$\bar{x} = \frac{\sigma \int_a^b x\, h(x)\, dx}{M}$$

$$\bar{y} = \frac{\sigma \int_c^d y\, w(y)\, dy}{M}.$$

Here $h(x)$ is the height of a vertical slice perpendicular to the x-axis and $w(y)$ the width of a horizontal slice perpendicular to the y-axis.

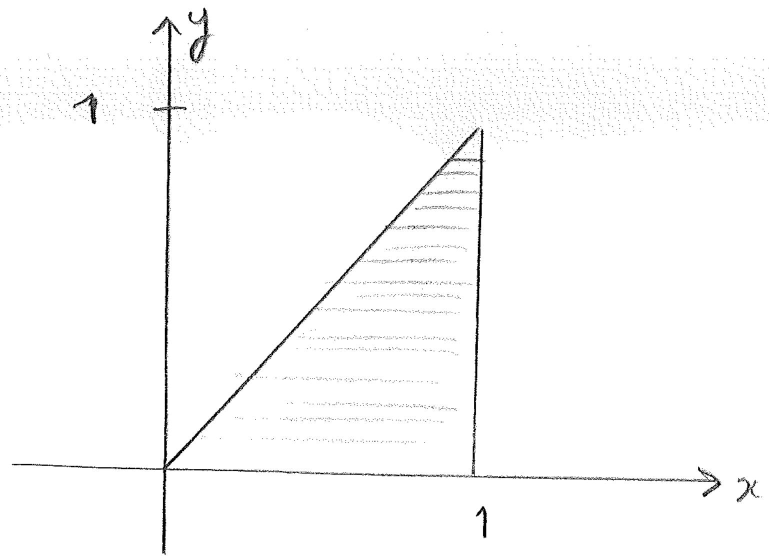


Of course, here

$M = \rho A$ where A is the area and

$$A = \int_a^b h(x)\, dx = \int_c^d w(y)\, dy.$$

Ex. Calculate the centre of mass of the uniform triangular lamina as shown.

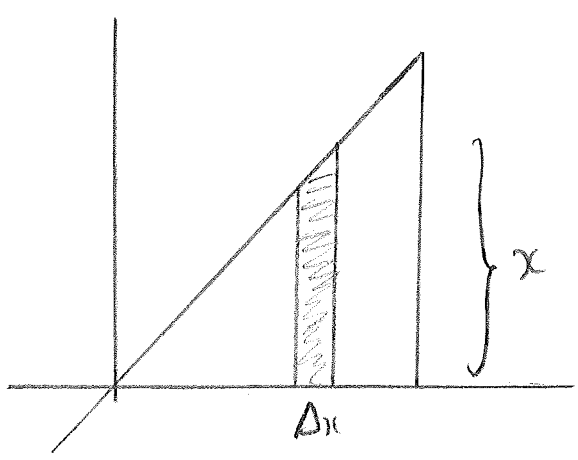


Let σ be the constant area density.

We can find the mass be 'cheating' since the lamina is uniform.

$$M = \sigma\,(\text{area}) = \sigma \times \tfrac{1}{2}\, 1 \times 1 = \frac{\sigma}{2}.$$

For $\bar{x}$ $\delta(x)$ is given by



$$\delta(x) = \sigma \times \text{length of small strip}$$
$$\approx \sigma x$$

Thus

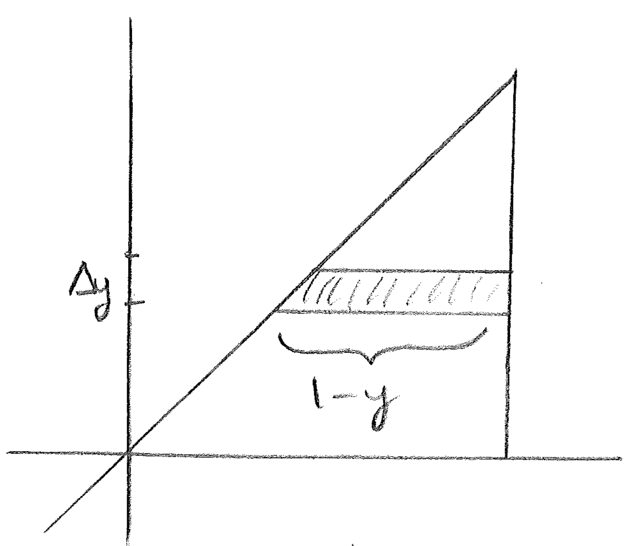
$$\int_a^b x\,\delta(x)\,dx = \int_0^1 x \cdot \sigma x\,dx$$

$$= \sigma \int_0^1 x^2\,dx$$

$$= \sigma \left[x^3/3 \right]_0^1$$

$$= \sigma/3 .$$

For $\bar{y}$, $\delta(y)$ is given by



$$\delta(y) = \sigma \times \text{length of small strip}$$

$$\approx \sigma(1-y)$$

Thus $\displaystyle\int_a^b y\,\delta(y)\,dy = \int_0^1 y\sigma(1-y)\,dy$

$$= \sigma \int_0^1 (y - y^2)\,dy$$

$$= \sigma \left[\frac{y^2}{2} - \frac{y^3}{3} \right]_0^1$$

$$= \sigma \left(\frac{1}{2} - \frac{1}{3} - 0 \right)$$

$$= \frac{\sigma}{6}.$$

Finally $\quad \bar{x} = \dfrac{\int_a^b x\,\delta(x)\,dx}{M} = \dfrac{\frac{\sigma}{3}}{\frac{\sigma}{2}} = \dfrac{2}{3}$

$$\bar{y} = \frac{\int_a^b y\,\delta(y)\,dy}{M} = \frac{\frac{\sigma}{6}}{\frac{\sigma}{2}} = \frac{1}{3}.$$

Three - Dimensional Regions.

For a 3-D region which lies between

$a \leq x \leq b, \; c \leq y \leq d, \; e \leq z \leq f,$

the centre of mass is given by

$$\bar{x} = \frac{\int_a^b x\,\delta(x)\,dx}{\int_o^b \delta(x)\,dx} = \frac{\int_a^b x\,\delta(x)}{M}$$

$$\bar{y} = \frac{\int_c^d y\,\delta(y)\,dy}{\int_c^d \delta(y)\,dy} = \frac{\int_c^d y\,\delta(y)}{M}$$

$$\bar{z} = \frac{\int_e^f z\,\delta(z)\,dz}{\int_e^f \delta(z)\,dz} = \frac{\int_e^f \delta(z)\,dz}{M}$$

Where $\delta(x), \delta(y), \delta(z)$ are the mass per unit length in the x-, y- and z-directions.

An important special case is when we have a constant uniform density, ρ.
In this case

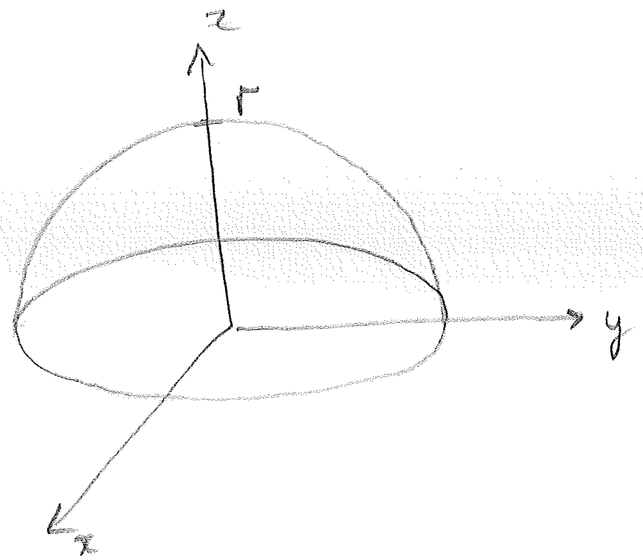
$$\bar{x} = \frac{\rho \int_a^b x A(x)\,dx}{M}$$

$$\bar{y} = \frac{\rho \int_c^d y A(y)\,dy}{M}$$

$$\bar{z} = \frac{\rho \int_e^f z A(z)\,dz}{M}$$

where $A(x), A(y), A(z)$ are the cross-sectional areas obtained by slicing perpendicular to the x-, y- and z- axes.

Ex. A uniform hemisphere of radius r.



If we place the hemisphere as shown, then by symmetry $\bar{x} = \bar{y} = 0$, so we only need to find $\bar{z}$.

Again, since the density is uniform, it is easy to find the mass, namely

$$\text{mass} = \text{density} \times \text{volume}$$

$$= \rho \times \frac{2}{3}\pi r^3$$

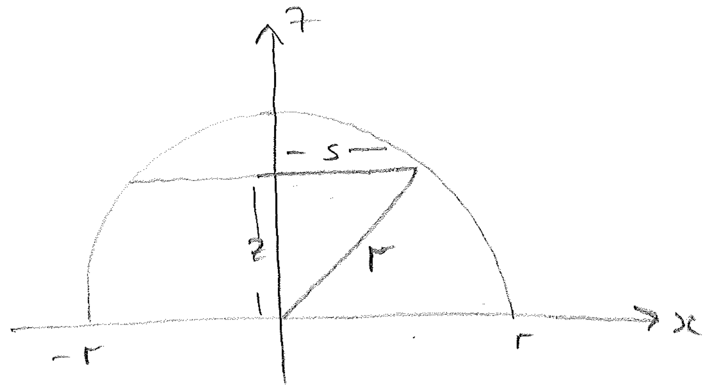
$$= \frac{2\pi r^3 \rho}{3}.$$

So we only need to find

$$\rho \int_e^f z A(z) dz .$$

The cross-section at height z is a disc of radius $s(z)$ where

$$s(z) = \sqrt{r^2 - z^2}.$$



By Pythagoras $r^2 = s^2 + z^2$

$$s^2 = r^2 - z^2$$

$$s = \sqrt{r^2 - z^2}$$

The area of this disc is then

$$A(z) = \pi (s(z))^2$$

$$= \pi \left(\sqrt{r^2 - z^2}\right)^2$$

$$= \pi (r^2 - z^2)$$

Thus

$$\rho \int_e^f z A(z) dz = \rho \int_0^r z \pi (r^2 - z^2) dz$$

$$= \pi \rho \int_0^r (r^2 z - z^3) dz$$

$$= \pi\rho\left[\frac{r^2z^2}{2} - \frac{z^4}{4}\right]_0^r$$

$$= \pi\rho\left(\frac{r^4}{2} - \frac{r^4}{4} - 0\right)$$

$$= \frac{\pi r^4 \rho}{4}.$$

Finally $\bar{z} = \dfrac{\int_0^r z A(z)\,dz}{M}$

$$= \frac{\cancel{\pi} r^4 \cancel{\rho} / 4}{2\cancel{\pi} r^3 \cancel{\rho} / 3}$$

$$= \frac{r^4/4}{2/3\, r^3}$$

$$= r \cdot \frac{1}{4} \times \frac{3}{2}$$

$$= \frac{3r}{8}.$$

Qs. Does this answer have the right units?

Does it surprise you that ρ doesn't appear in the answer?

Is the answer about the size you'd expect?