§ 11.6 Applications and Modelling

<u>Ex</u> A Frozen Lake.

Ice forms on the surface of a lake and thickens over time. However, the thicker the ice, the less heat makes it through the ice layer and the slower the thickness of the ice increases.

Expect the thickness of the ice with time to be <u>increasing</u> and <u>concave down</u>.

A reasonable guess

$$\text{Rate thickness is increasing} = \frac{k}{\text{thickness}}$$

for some constant k (ie rate at which ice forms is inversely proportional to thickness).

So if we let $y = y(t)$ be the thickness as a fn of time t, we have

$$\frac{dy}{dt} = \frac{k}{y}, \qquad k > 0, \; y > 0.$$

Separate variables and integrate

$$\int y \, dy = \int k \, dt$$

$$\frac{y^2}{2} = kt + C$$

If we initially start with no ice, $y(0) = 0$ and

$$\frac{0^2}{2} = k \cdot 0 + C$$

$$\Rightarrow \quad C = 0$$

and the sol^n of the IVP

$$\frac{dy}{dt} = \frac{k}{y}, \quad y(0) = 0 \quad \text{is}$$

$$\frac{y^2}{2} = kt$$

ie $\quad y = \sqrt{2kt} \quad$ (remember, we want $y > 0$).

Rem The bigger k is, the faster the ice forms.

Ex. Net Worth of a Company.

A company's revenue is earned at a continuous rate of 5% of its net worth. Hower, the company must pay out \$200 million each year in payroll.

Assuming these two factors are the only ones affecting the company's worth

a) Give a DE which governs the net worth W in millions of dollars.

b) Solve the DE to find W, given an initial value of W_0 million dollars.

c) Sketch the soln for

$W_0 = 3000, 4000, 5000.$

Solution Note first that, if the company's worth stays constant at W_0, then

Rate revenue is earned $=$ Rate payments are made

Since the revenue $=$ 5% net worth $= .05 W_0$, we have

$$.05 W_0 = 200$$

$$\Rightarrow W_0 = 4000 \qquad (= 4 \text{ billion } \$),$$

So if $W_0 = 4{,}000$ revenue balances payments exactly.

If $W_0 > 4{,}000$, revenue exceeds payments and company makes more and more money.

If $W_0 < 4{,}000$, payments exceed revenue and company loses money, eventually going bankrupt (as we'll see).

a) Set up the DE using

Rate net worth is increasing = Rate revenue is earned − Rate payroll payments are made.

If the net worth is W, then revenue is earned at a rate of $.05W$ while payments are made at a rate of 200/yr so that

$$\frac{dW}{dt} = 0.05W - 200.$$

b) For convenience, we first rewrite as

$$\frac{dW}{dt} = 0.05(W - 4{,}000)$$

(note how this shows how $\frac{dW}{dt} = 0$ if $W = W_0 = 4{,}000$ as we already saw).

Now separate variables and integrate

$$\int \frac{dW}{W-4{,}000} = \int 0.05\,dt$$

$$\Rightarrow \quad \ln|W-4{,}000| = 0.05t + C,$$

C const.

Take exp of both sides (as usual),

$$|W-4{,}000| = e^{0.05t+C}$$

$$= e^{C} \cdot e^{0.05t}$$

$$\Rightarrow \quad W - 4{,}000 = \pm e^{C} \cdot e^{0.05t}$$

$$= A e^{0.05t}$$

where $A = \pm e^{C}$ is another constant.

So

$$W = 4{,}000 + A e^{0.05t}.$$

To find A, we use the IC $W(0) = W_0$.

$$W_0 = 4{,}000 + Ae^0 = 4{,}000 + A$$

$$\Rightarrow \quad A = W_0 - 4{,}000.$$

So the (unique) sol^{n} of the IVP is

$$W = 4{,}000 + (W_0 - 4{,}000)e^{0.05t}$$

c) If $W_0 = 4{,}000$, $W = 4{,}000$ always
- equilibrium solution

If $W_0 = 5{,}000$, $W = 4{,}000 + 1{,}000e^{0.05t}$
- net worth grows without bound

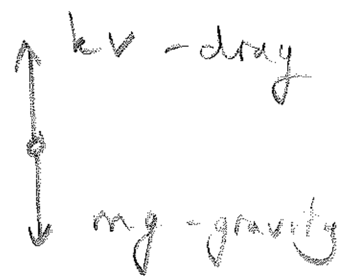
d) If $W_0 = 3{,}000$, $W = 4{,}000 - 1{,}000e^{0.05t}$

- net worth decreases and company goes bankrupt (in about 27.7 years).

Note how the above shows how the equilibrium of 4,000 is <u>unstable</u>.

Ex Falling Bodies - Terminal Velocity

A small body falling experiences acceleration due to gravity and drag proportional to its velocity, which gives a net force



$$F = mg - kv$$

(where m is of course the mass and v the velocity).

By Newton's second law

$$F = ma = m\frac{dv}{dt}$$

and so we obtain the DE

$$m\frac{dv}{dt} = mg - kv$$

ie

$$\frac{dv}{dt} = g - \frac{k}{m}v.$$

Again, for the purpose of separating the variables, it is convenient to factor out $\frac{k}{m}$ on the right to get

$$\frac{dv}{dt} = \frac{k}{m}\left(\frac{mg}{k} - v\right)$$

or,

$$\frac{dv}{dt} = -\frac{k}{m}\left(v - \frac{mg}{k}\right)$$

Separate and integrate

$$\int \frac{dv}{v - \frac{mg}{k}} = -\frac{k}{m}\int dt$$

$$\ln\left|v - \frac{mg}{k}\right| = -\frac{k}{m}t + C, \quad C \text{ const.}$$

Take exp of both sides

$$\left|v - \frac{mg}{k}\right| = e^{-\frac{kt}{m} + C} = e^{C} \cdot e^{-\frac{kt}{m}}$$

$$v - \frac{mg}{k} = Ae^{-\frac{kt}{m}}, \quad A = \pm e^{c}_{\text{const}}$$

$$v = \frac{mg}{k} + Ae^{-\frac{kt}{m}}.$$

We can find A using an initial condition. If the body starts from rest, $v(0) = 0$ and so

$$0 = \frac{mg}{k} + Ae^{0}$$

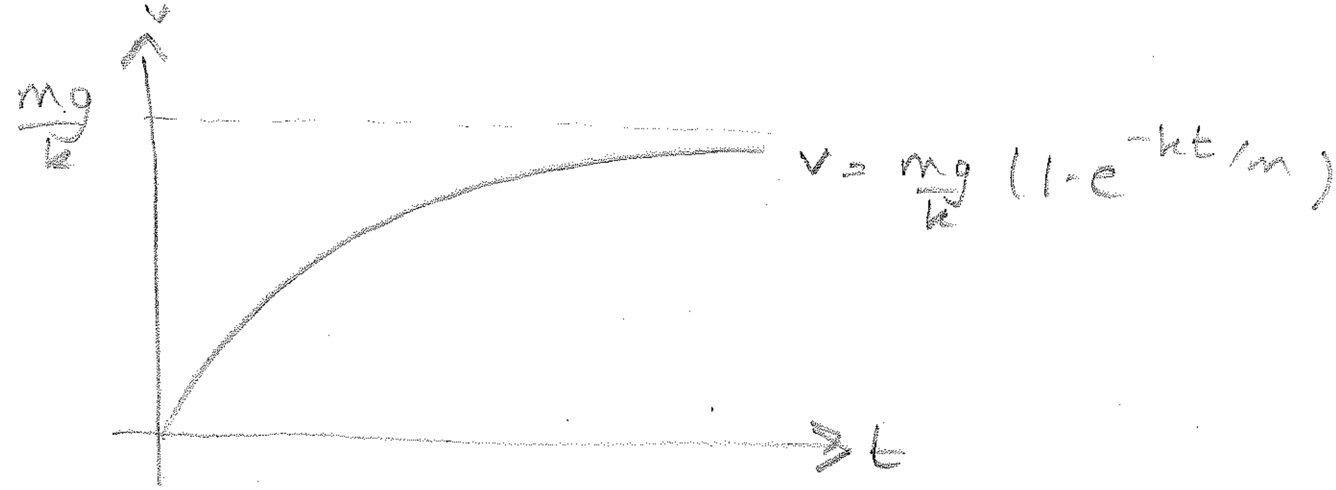
$$\Rightarrow \quad A = -\frac{mg}{k}$$

and

$$v = \frac{mg}{k} - \frac{mg}{k}e^{-kt/m}$$

$$= \frac{mg}{k}\left(1 - e^{-kt/m}\right).$$

Graph of velocity looks like



Note how the vel. tends asymptotically to the <u>terminal velocity</u> $\frac{mg}{k}$ as $t \to \infty$.

Can also get the terminal vel. directly from the DE by setting $\frac{dv}{dt} = 0$ and solving for v:

$$m\frac{dv}{dt} = mg - kv = 0$$

$$\Rightarrow \quad mg = kv$$

$$v = \frac{mg}{k}.$$

Ex. Salt in the Scituate reservoir.

The reservoir in Scituate, RI holds 100 million gallons of H_2O. and supplies Providence with 1 million gallons per day. The reservoir is partly refilled by springs which give 0.9 million gallons/day and the remaining 0.1 million gallons/day to keep the volume constant comes from runoff from the surrounding land.

If the runoff contains salt dissolved with a concentration of 0.0001 lbs/gallon and the reservoir initially contains no salt, find the concentration of salt as a function of time (assuming the reservoir is well-mixed so that the concentration is uniform).

Solution Let $C = C(t)$ be the concentration (in lbs/gal) of NaCl and let $Q = Q(t)$ be the total quantity of NaCl in the reservoir (in lbs).

C and Q are related by

$$C = \frac{Q}{10^8} \quad \left(\text{concentration} = \frac{\text{quantity}}{\text{volume}} \right).$$

We'll find Q first and then use the above to get C.

We have that

Rate of change of quantity of NaCl = Rate NaCl enters

− Rate NaCl leaves.

Salt enters the reservoir through the 0.1 million gal of runoff with a conc. of 10^{-4} lbs/gal.

$$\text{Rate salt enters} = \text{concentration} \times \text{volume per day}$$

$$= 10^{-4} \text{ lbs/gal} \times 10^{5} \text{ gal/day}$$

$$= 10 \text{ lbs/day}$$

Otoh, NaCl leaves dissolved at a conc. of C lbs/gal in the 10^6 gal. consumed by Providence.

$$\text{Rate salt leaves} = \text{Concentration} \times \text{volume per day}$$

$$= C \ \tfrac{\text{lbs}}{\text{gal}} \times 10^{6} \text{ gal}$$

$$= \frac{Q}{10^{8}} \times 10^{6} = \frac{Q}{100} \text{ lbs/day}$$

Putting everything together

$$\frac{dQ}{dt} = 10 - \frac{Q}{100}$$

$$= -.01(Q - 1{,}000)$$

Separate and integrate

$$\int \frac{dQ}{Q - 1{,}000} = -\int 0.01\,dt$$

$$\ln|Q - 1{,}000| = -0.01t + C, \quad C \text{ const}$$

Take exp. of both sides.

$$|Q - 1{,}000| = e^{-0.01t + c} = e^{c} \cdot e^{0.01t}$$

$$Q - 1{,}000 = Ae^{-0.01t}, \quad A = \pm e^{c}$$

const

$$Q = 1{,}000 + Ae^{-0.01t}$$

To find A use the IC that there is no salt initially so that $Q(0)=0$

$$0 = 1{,}000 + Ae^0$$

$$\Rightarrow \quad A = -1{,}000$$

and $Q = 1{,}000 - 1{,}000\, e^{-0.01t}$

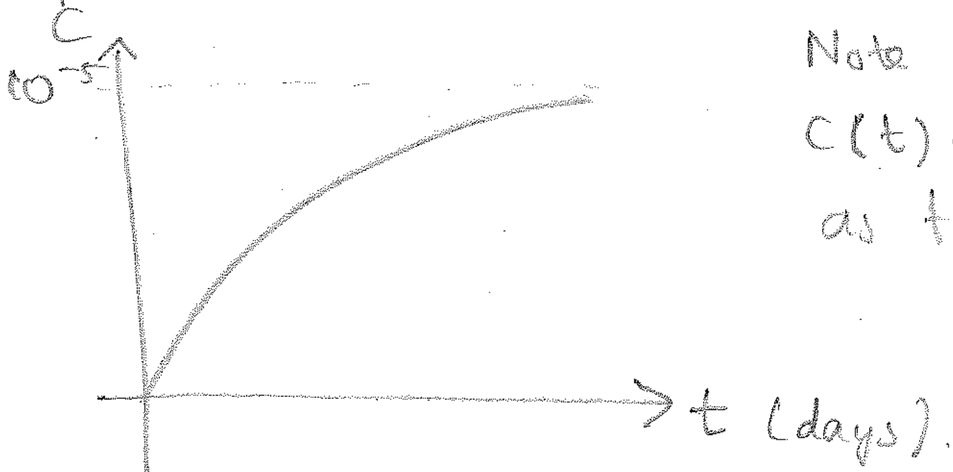
$$= 1{,}000\,(1 - e^{-0.01t}) \text{ lbs},$$

Finally

$$C = \frac{Q}{10^8} = 10^{-8} \times 10^3\,(1 - e^{-0.01t})$$

$$= 10^{-5}\,(1 - e^{-0.01t}) \text{ lbs/gal}.$$

Graph looks like



Note how $C(t) \to 10^{-5}$ lbs/gal as $t \to \infty$.