Chapter 11 Differential Equations

§ 11.1 Introduction

A <u>differential equation</u> (DE) in a fn $y = y(x)$ is an equation involving derivatives of y (possibly including y itself) and (possibly) x.

e.g.

$$y' = x \sin y$$

$$y'' + y' + 3y = 0$$

$$\sin(y''') + y' = 7\sqrt{4 - x^2}.$$

The <u>order</u> of a DE is the power of the highest derivative appearing. The first eqn above is first order, the second second order and the third third order.

The goal with a DE is to find as many fns $y(x)$ as possible which satisfy the DE. Such fns are called <u>solutions</u> of the DE.

<u>Ex</u> For the DE

$$y' = x^2$$

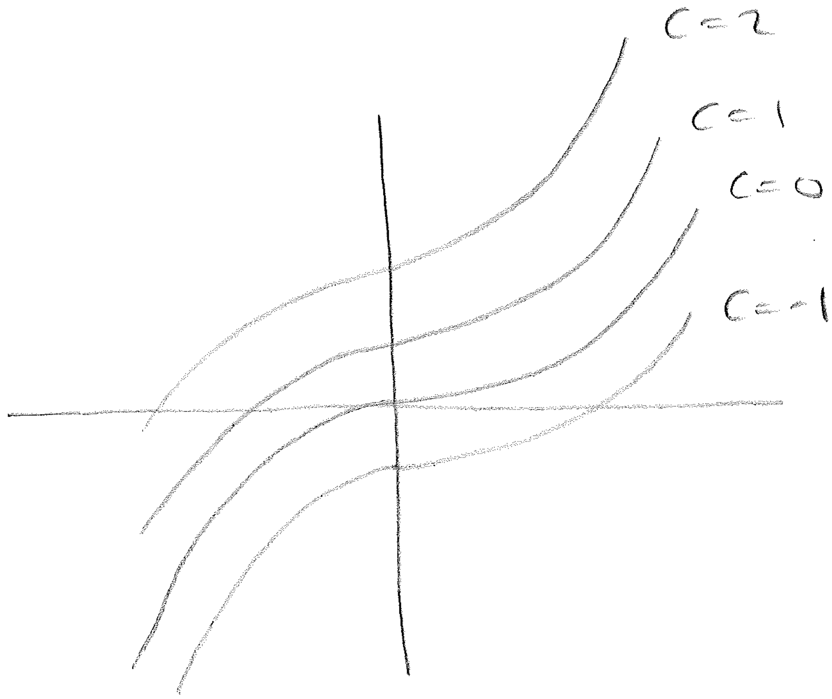
if we take antiderivatives of both sides

$$\int y'(x)\,dx = \int x^2\,dx$$

$$\Rightarrow \quad y(x) = \frac{x^3}{3} + C$$

where C is an arbitrary constant which arises due to the fact that a given fn has many antiderivatives all of which differ by a constant.

Allowing C to vary over all real numbers gives us all possible solns of this DE.



If we want to pick a particular soln, we can do this by requiring that our soln has a specified value at a certain point.

e.g. If we require that $y(0) = 1$, then

$$1 = \frac{0^3}{3} + c$$

$$\Rightarrow \quad c = 1$$

and so

$$y = \frac{x^3}{3} + 1$$

is the unique soln satisfying $y(0) = 1$.

A problem of this type is called an initial value problem (IVP) and the condition $y(0) = 1$ is called an initial condition (IC).

Ex. Solve the seccond order DE

$$y'' = -\cos x$$

Taking antids once gives

$$y' = -\sin x + C_1 .$$

This only gives us an eqn in y', so we need to take antids. again to get.

$$y = \cos x + C_1 x + C_2 .$$

Note that this time we have two arbitrary constants.

This is a general feature, namely that the number of constants in a solution is the same as the order of the DE.

If we want to find a particular sol^n using initial conditions for the last example, we need two pieces of information, one for each constant.

e.g. If we impose the ICs $y(0) = 2$, $y'(0) = 1$

we get

$$y(0) = 2 = \cos 0 + C_1 \cdot 0 + C_2$$

$$2 = 1 + C_2$$

$$\Rightarrow C_2 = 1.$$

$$y'(x) = -\sin x + C_1$$ (as we saw earlier).

So $y'(0) = 1 = -\sin 0 + C_1$

$\Rightarrow \quad 1 = C_1$

Our unique sol^n to the IVP

$$y' = -\cos x, \qquad y(0) = 2,\ y'(0) = 1$$

is then

$$y = \cos x + x + 1.$$