§ 8.2 Applications to Geometry

Already saw a number of examples of how to use integration to find area/volume.

In general, the approach is as follows.

To Compute a Volume/Area/Length Using an Integral

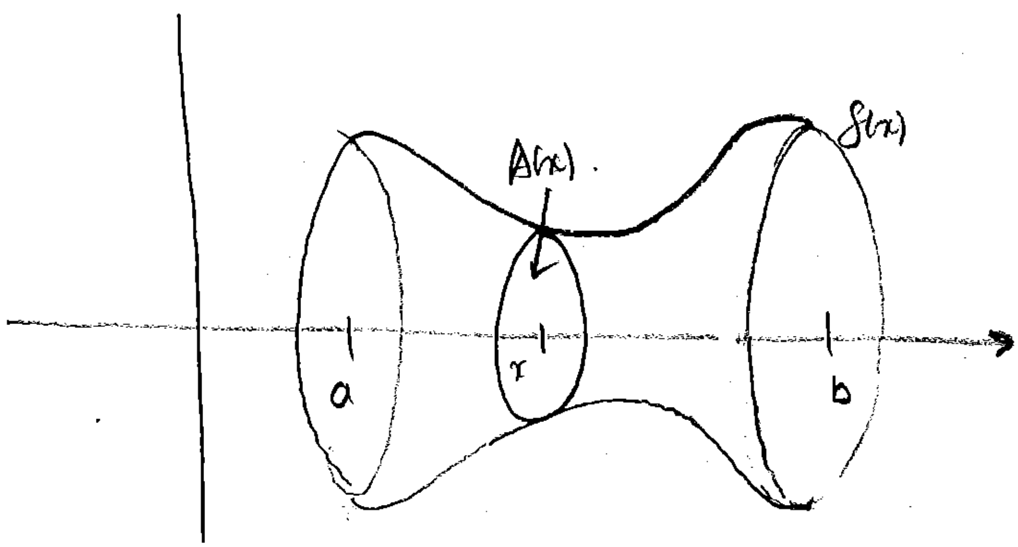
- Divide the solid/region/curve into small pieces whose volume/area/length we can easily approximate.

- Add the contributions of the pieces together - gives us a Riemann sum which approximates the total volume/area/length.

- Take the limit as the number of terms in the sum tends to infinity. Leads to a definite integral for the total volume/area/length.

Volumes of Revolution

Suppose we take a fn $f(x)$ which is $\geqslant 0$ on $[a, b]$ and rotate it about the axis. The curve sweeps out a solid with round cross-sections (like a lathe). Called a <u>volume of revolution</u>.

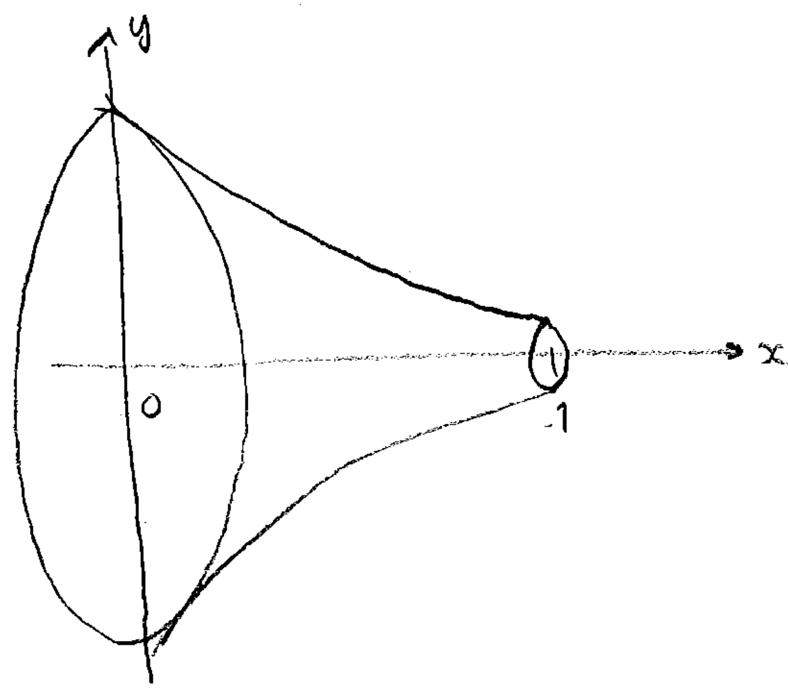


Here $A(x) = \pi (f(x))^2$, so

$$V = \int_a^b A(x)\,dx = \pi \int_a^b (f(x))^2\,dx = \pi \int_a^b (r(x))^2\,dx$$

Similar formulae exist for rotating about other lines (e.g. y-axis, $x = 3$).

Ex. The curve $y = e^{-x}$, $0 \le x \le 1$ is rotated about the x-axis.



Here $a = 0$, $b = 1$, $f(x) = e^{-x}$ and

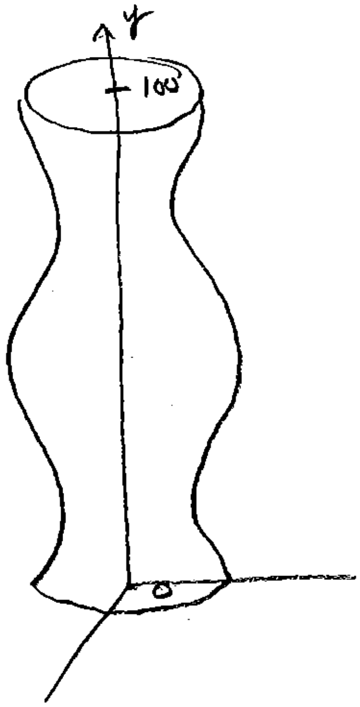
$$V = \pi \int_0^1 (f(x))^2 \, dx$$

$$= \pi \int_0^1 (e^{-x})^2 \, dx$$

$$= \pi \int_0^1 e^{-2x} \, dx = \pi \left[-\frac{1}{2} e^{-2x} \right]_0^1 = \pi \left(-\frac{1}{2} e^{-2} - \left(-\frac{1}{2} \cdot 1 \right) \right)$$

$$= \frac{\pi}{2} (1 - e^{-2}) \approx 1.36$$

Ex. Table leg is obtained by rotating $r = 3 + \cos(\pi y/25)$ cm $0 \le y \le 100$ cm about the y-axis.



$$V = \pi \int_0^{100} (r(y))^2 \, dy$$

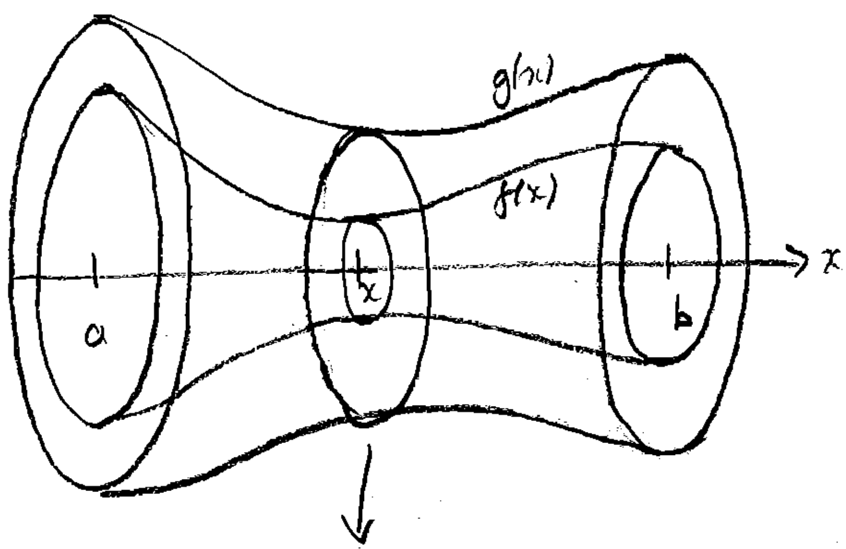
$$= \pi \int_0^{100} (3 + \cos(\pi y/25))^2 \, dy$$

$$= \pi \int_0^{100} (9 + 6\cos(\pi y/25) + \cos^2(\pi y/25)) \, dy$$

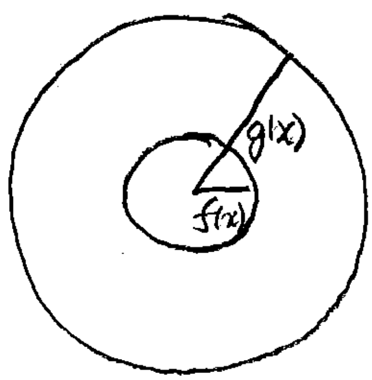
$$\approx 2984.5 \text{ cm}^3.$$

Check the (boring) details of the integration as an exercise!

If $0 \leq f(x) \leq g(x)$ on $[a,b]$ and we rotate the region between these curves about the x-axis, we have a volume whose cross-sections are annuli (rings).



Annulus.



Here

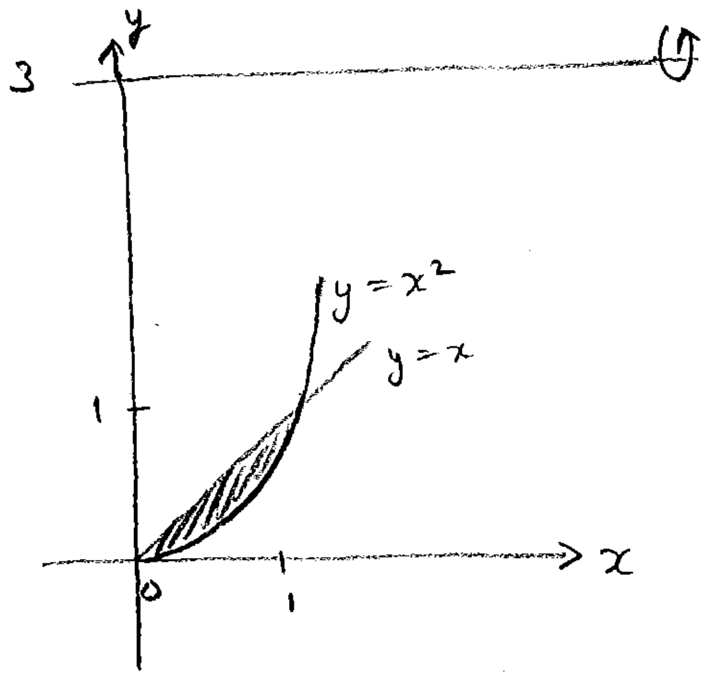
$$A(x) = \pi (g(x))^2 - \pi (f(x))^2$$
$$= \pi \left((g(x))^2 - (f(x))^2 \right)$$

Then

$$V = \pi \int_a^b \left((g(x))^2 - (f(x))^2 \right) dx$$

Again, similar formulae exist for rotating about other lines.

Ex 3. Region enclosed between $y = x$ and $y = x^2$ is rotated about the horizontal line $y = 3$.



Need to intersect the two curves to determine the limits of integration a, b.

So set

$$x = x^2$$

$$x - x^2 = 0$$

$$x(1-x) = 0$$

$$x = 0, 1$$

So $a = 0, \ b = 1$.

Also $y = x^2$ is clearly the curve which is <u>furthest from the axis of rotation</u>

(e.g. $3-(\frac{1}{2})^2 > 3-\frac{1}{2}$), so here

$$f(x) = 3 - x$$

$$g(x) = 3 - x^2.$$

$$V = \pi \int_0^1 ((3-x^2)^2 - (3-x))^2 dx$$

$$= \pi \int_0^1 (9 - 6x^2 + x^4 - (9 - 6x + x^2)) dx$$

$$= \pi \int_0^1 (6x - 7x^2 + x^4) dx$$

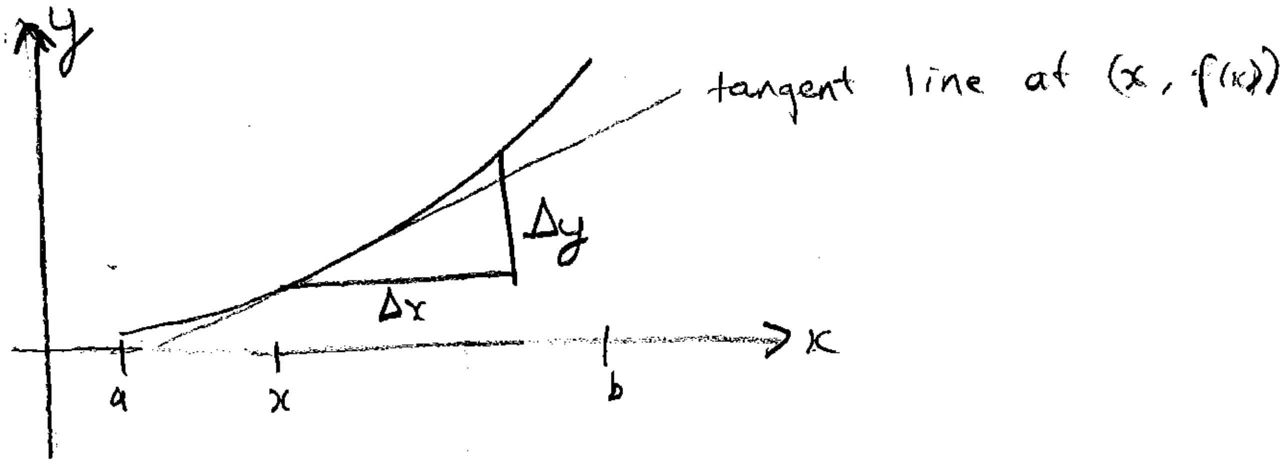
$$= \pi \left[3x^2 - \frac{7x^3}{3} + \frac{x^5}{5} \right]_0^1$$

$$= \pi \left(3 - \frac{7}{3} + \frac{1}{5} - 0 \right)$$

$$= \pi \left(\frac{45 - 35 + 1}{15} \right) = \frac{11\pi}{15}$$

Arc Length

Suppose we want to find the length, s, of the curve $y = f(x)$, $a \leq x \leq b$



If we approximate a small segment of the curve by a line segment, we get that the length of this part of the curve is, using Pythageras, approx

$$\Delta s \approx \sqrt{(\Delta x)^2 + (\Delta y)^2}$$

Using the tangent line approximation (§ 3.9),

$$\Delta y \approx f'(x)\,\Delta x.$$

Thus

$$\Delta s \approx \sqrt{(\Delta x)^2 + (f'(x))^2 (\Delta x)^2}$$

$$= \sqrt{(1 + (f'(x))^2)(\Delta x)^2}$$

$$= \sqrt{(1 + (f'(x))^2} \, \Delta x.$$

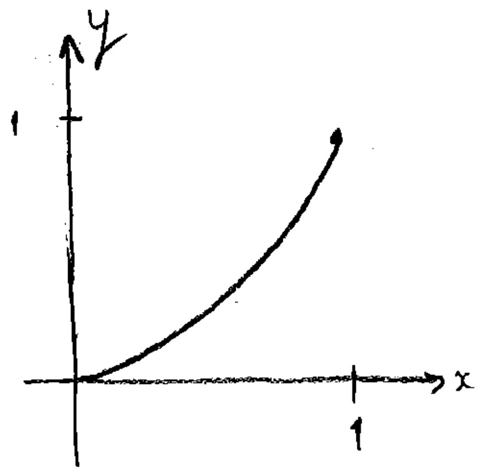
This leads to a Riemann sum

$$s \approx \sum_{i=1}^{n} \sqrt{(1 + (f'(x_{i-1}))^2} \, \Delta x.$$

Taking limits as $n \to \infty$, we <u>define</u> the arc length s by

$$s = \int_a^b \sqrt{1 + (f'(x))^2} \, dx.$$

<u>Ex</u> $\quad f(x) = \frac{2}{3}x^{3/2}, \quad 0 \le x \le 1$



$$f'(x) = x^{\frac{1}{2}}$$

$$\sqrt{1+(f'(x))^2} = \sqrt{1+(\sqrt{x})^2} = \sqrt{1+x}.$$

Thus

$$S = \int_0^1 \sqrt{1+(f'(x))^2}\,dx$$

$$= \int_0^1 \sqrt{1+x}\,dx$$

[If you like, let
$w = 1+x, \quad dw = dx$
when $x=0, w=1, \; x=1, w=2$]

$$= \left[\frac{2}{3}(1+x)^{3/2}\right]_0^1$$

$$= \frac{2}{3}\left(2^{3/2}-1\right).$$

Note: It is quite hard to find examples for arc-length where $\sqrt{1+(f'(x))^2}$ has an elementary antiderivative.