§ 11.5 Growth and Decay

In the last section we saw that the IVP

$$y' = ky, \quad y(0) = y_0$$

has the (unique) sol$^{\underline{n}}$

$$y(t) = y_0 e^{kt}.$$

Thus, if we have a population $P = P(t)$ with initial value P_0 whose growth rate is proportional to the size of the population at that time, then we have

$$P = P_0 e^{kt}.$$

As we saw earlier, $k > 0$ gives us exponential growth and $k < 0$ gives us exponential decay.

Ex. Continuously Compounded Interest

Suppose I invest B_0 dollars in a bank account with a continuously compounded interest rate of 5%.

This means that

Rate at which balance is growing = 5% (Size of balance)

$$\frac{dB}{dt} = .05\,B \qquad (5\% = .05).$$

This gives

$$B = B_0\, e^{0.05t}$$

Using this formula, we can answer other questions.

e.g. suppose I want to know how long it'll take me to triple my investment.

If I triple my money, then I have $3B_0$ dollars (B_0 being the initial investment).

I then set

$$3B_0 = B_0 e^{0.05t}$$

$$3 = e^{0.05t}$$

and solve for t in the usual way by taking the ln of both sides

$$\ln 3 = \ln(e^{0.05t})$$

$$\ln 3 = 0.05t$$

$$t = \frac{\ln 3}{0.05} = 20 \ln 3 \approx 21.97 \text{ years.}$$

N.b. note the difference between

Continuously compounded interst at 5%

$$B = B_0 e^{0.05t}$$

Annually compounded interest at 5%

$$B = B_0 (1.05)^t.$$

Q. Which scheme gives a better return on your investment?

Ex. Newton's Law of Cooling

This states that if we place a body in surroundings at a certain temperature then the body's temperature changes to match that of the surroundings at a rate proportional to the difference in temperature between the body and the surroundings.

Ex. Suppose a murder is committed on a day when the temp. is a constant 20°C. Suppose also that the body is initially at 37°C and that after 2 hours it is at 35°C.

a) Find the temp., $H = H(t)$ of the body as a f$^{\underline{n}}$ of the time t in hours since the murder was committed.

b) Sketch a graph of temp. against time.

c) What happens to the temp. in the long run?

d) If the body is found at 4pm at a temp of $30°C$, when was the murder committed?

a) By Newton's law of cooling, for some constant α we have that

Rate of change of temp $= \alpha$ (temp. difference).

If $H = H(t)$ is the temp, then $H - 20$ is the temp difference, and so

$$\frac{dH}{dt} = \alpha(H-20).$$

Note that if $H > 20$, then $\frac{dH}{dt}$ should be <u>negative</u> (and if $H < 20$ $\frac{dH}{dt}$ should be <u>positive</u>).

This means that α should be negative,
so if we write $\alpha = -k$ $(k > 0)$, then

$$\frac{dH}{dt} = -k(H-20).$$

We solved this DE in the last section by separation of variables to get

$$H = 20 + Be^{-kt}$$

To find B we use the IC $H(0) = 37$.

$$37 = 20 + Be^{-k \cdot 0}$$

$$37 = 20 + B$$

$$B = 17.$$

Thus $H = 20 + 17e^{-kt}$.

We're not done yet as we still need to find k.

To do this we use the fact that the body was at 35°C after 2 hours.

$$35 = 20 + 17e^{-k \cdot 2}$$

$$15 = 17e^{-2k}$$

$$\frac{15}{17} = e^{-2k}$$

Now take ln of both sides

$$\ln\left(\frac{15}{17}\right) = \ln\left(e^{-2k}\right)$$

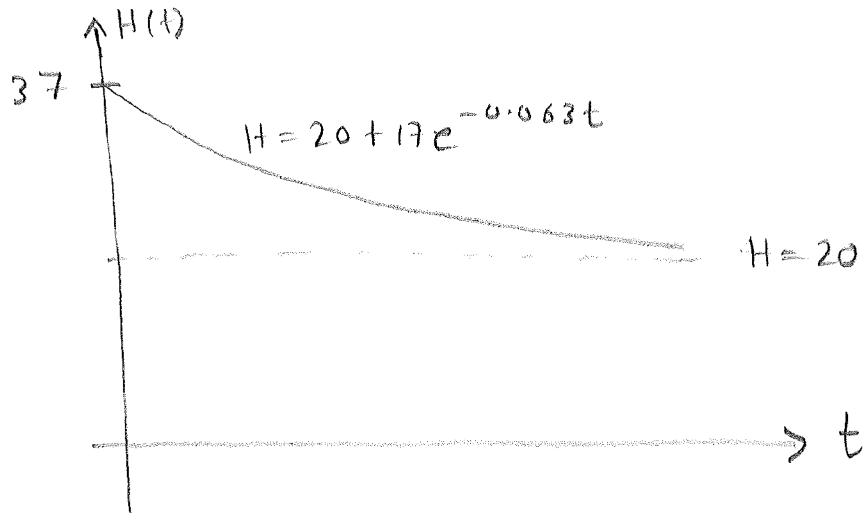
$$\ln\left(\frac{15}{17}\right) = -2k$$

$$k = \frac{-\ln\left(\frac{15}{17}\right)}{2} \approx 0.063.$$

The answer for a) is then

$$H = 20 + 17e^{-0.063t}$$

b) The graph of H looks as follows



Note that the y-intercept is 37 as $H(0) = 37$.

c) 'In the long run' means as $t \to \infty$.

From the graph it appears that $H \to 20$ as $t \to \infty$.

More rigorously

$$\lim_{t\to\infty} H(t) = \lim_{t\to\infty} (20+17e^{-0.063t})$$

$$= 20 + 17\lim_{t\to\infty} e^{-0.063t}$$

$$= 20 + 17(0)$$

$$= 20° C.$$

d) We want do know when $H = 30$, so we set

$$30 = 20 + 17e^{-0.063t}$$

and solve for t.

$$10 = 17e^{-0.063t}$$

$$\frac{10}{17} = e^{-0.063t}$$

Take ln of both sides

$$\ln\left(\frac{10}{17}\right) = \ln\left(e^{-0.063t}\right)$$

$$\ln\left(\frac{10}{17}\right) = -0.063t$$

$$t = -\frac{\ln\left(\frac{10}{17}\right)}{0.063} \approx 8.4 \text{ hrs.}$$

This means that if the body was found at 4pm, then the murder was committed about 8 hrs, 24 min earlier, i.e. just after 7:30 am.

In the last example, since

$$\frac{dH}{dt} = -k(20-H)$$

If H is the constant fn 20, then

$\frac{dH}{dt}$ and $20-H$ are both 0.

Thus both sides of the DE are 0 and so equal which means that $H = 20$ is a solution.

A constant solution of this type is called an equilibrium solution (or just equilibrium).

It is an example of a <u>stable equilibrium</u> which is an equilibrium sol^n to which other sol^{n}s with nearby initial conditions converge.

There are also <u>unstable equilibrium solutions</u>.

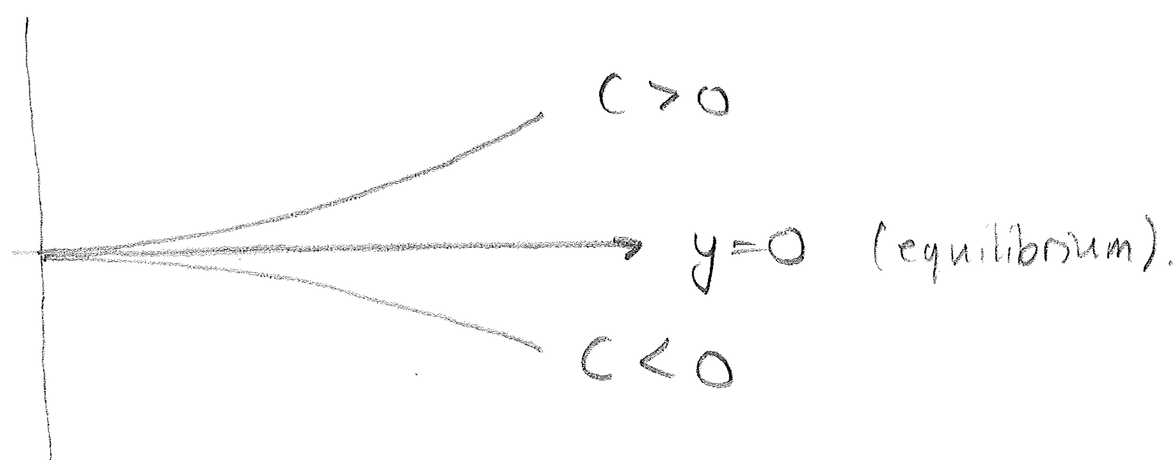
These are equilibrium sol^ns from which sol^ns with nearby initial conditions diverge.

e.g. $$\frac{dy}{dt} = y$$

has sol^ns

$$y = Ce^{t}, \quad C \text{ constant}.$$

$y=0$ is an equilibrium sol^n which is unstable as can be seen by looking at the following picture.



Conceptually a stable equilibrium is like a marble at the bottom of a valley while an unstable equilibrium is like a marble on a hilltop.