Worksheet #9

1. **(a)** $\sum_{n=0}^{\infty}\frac{2^n \cdot x^{2n}}{(2n)!}$

$$\lim_{n\to\infty}\frac{|a_{n+1}|}{|a_n|}=\lim_{n\to\infty}\frac{\left|\frac{2^{n+1}\cdot x^{2(n+1)}}{(2(n+1))!}\right|}{\left|\frac{2^n\cdot x^{2n}}{(2n)!}\right|}=\lim_{n\to\infty}\frac{2\cdot x^2}{(2n+2)(2n+1)}=0$$

So the radius of convergence is infinite.

(b) $\sum_{n=0}^{\infty}\frac{4^n\cdot(n!)^2\cdot(x-1)^n}{(2n)!}$

$$\lim_{n\to\infty}\frac{|a_{n+1}|}{|a_n|}=\lim_{n\to\infty}\frac{\left|\frac{4^{n+1}\cdot((n+1)!)^2\cdot(x-1)^{n+1}}{(2(n+1))!}\right|}{\left|\frac{4^n\cdot(n!)^2\cdot(x-1)^n}{(2n)!}\right|}=\lim_{n\to\infty}\frac{4\cdot(n+1)^2\cdot|x-1|}{(2n+2)(2n+1)}=1\cdot|x-1|$$

The radius of convergence is 1.

2. **(a)** For $n = 4$, $a = 0$ and $f(x) = \cos x$,

$f(a)=\cos a=\cos 0=1$

$f'(a)=-\sin a=-\sin 0=0$

$f''(a)=-\cos a=-\cos 0=-1$

$f'''(a)=\sin a=\sin 0=0$

$f^{(4)}(a)=\cos a=\cos 0=1$

$$f(x)=\frac{f(a)}{0!}(x-a)^0+\frac{f'(a)}{1!}(x-a)^1+\frac{f''(a)}{2!}(x-a)^2+\ldots..$$

Thus, for x near 0,

$$\cos x\approx\frac{1}{0!}\cdot 1+\frac{0}{1!}\cdot(x-0)^1+\frac{-1}{2!}\cdot(x-0)^2+\frac{0}{3!}\cdot(x-0)^3+\frac{1}{4!}\cdot(x-0)^4$$

$$=1-\frac{x^2}{2!}+\frac{x^4}{4!}$$

(b) For x near 1, $e^x\approx e+e(x-1)+\frac{e}{2!}(x-1)^2+\frac{e}{3!}(x-1)^3+\frac{e}{4!}(x-1)^4$

(c) For $n=4$, $a=0$, $f(x)=\sqrt[3]{1-x}$

$$f(a)=(1-a)^{\frac{1}{3}}=1$$

$$f'(a)=-\tfrac{1}{3}(1-a)^{-\frac{2}{3}}=-\tfrac{1}{3}$$

$$f''(a)=-\tfrac{2}{9}(1-a)^{-\frac{5}{3}}=-\tfrac{2}{9}$$

$$f'''(a)=-\tfrac{10}{27}(1-a)^{-\frac{8}{3}}=-\tfrac{10}{27}$$

$$f^{(4)}(a)=-\tfrac{80}{81}(1-a)^{-\frac{11}{3}}=-\tfrac{80}{81}$$

$$f(x)\approx 1-\tfrac{1}{3}x-\frac{\frac{2}{9}x^2}{2!}-\frac{\frac{10}{27}x^3}{3!}-\frac{\frac{80}{81}x^4}{4!}$$

$$=1-\tfrac{1}{3}x-\tfrac{1}{9}x^2-\tfrac{5}{9}x^3-\tfrac{10}{243}x^4$$

(d) For x near $\frac{\pi}{2}$, $\sin x\approx 1-\dfrac{\left(x-\frac{\pi}{2}\right)^2}{2!}+\dfrac{\left(x-\frac{\pi}{2}\right)^4}{4!}$.

(e) For x near 1, $\ln\left(x^2\right)\approx 0+2(x-1)-\dfrac{2(x-1)^2}{2!}+\dfrac{4(x-1)^3}{3!}$.

3. **(a)** $\sin x = x-\dfrac{x^3}{3!}+\dfrac{x^5}{5!}-\dfrac{x^7}{7!}+....+\dfrac{(-1)^{n+1}\cdot x^{2n-1}}{(2n-1)!}+.....$

$$\sin x=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\cdot x^{2n-1}}{(2n-1)!}$$

(b) $\arctan x = x-\dfrac{x^3}{3}+\dfrac{x^5}{5}-\dfrac{x^7}{7}+....+\dfrac{(-1)^{n+1}\cdot x^{2n-1}}{2n-1}+.....$

$$\arctan x=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\cdot x^{2n-1}}{2n-1}$$

(c) $e^{-x^2}=1-x^2+\dfrac{x^4}{2!}-\dfrac{x^6}{3!}+\dfrac{x^8}{4!}-....+\dfrac{(-1)^n\cdot x^{2n}}{n!}+.....$

$$e^{-x^2}=\sum_{n=0}^{\infty}\frac{(-1)^n\cdot x^{2n}}{n!}$$

4. Given: y is near 0. Change the problem slightly so that (c) is $\dfrac{1}{\sqrt{1-y^2}}-1$.

$$\ln\left(1+\tfrac{y^2}{2}\right)\approx\frac{\frac{y^2}{2}}{1}-\frac{\left(\frac{y^2}{2}\right)^2}{2}+\frac{\left(\frac{y^2}{2}\right)^3}{3}-\frac{\left(\frac{y^2}{2}\right)^4}{4}=\tfrac{1}{2}y^2-\tfrac{1}{8}y^4+\tfrac{1}{24}y^6-\tfrac{1}{64}y^8$$

$$1-\cos y\approx 1-\left(1-\frac{y^2}{2!}+\frac{y^4}{4!}-\frac{y^6}{6!}+\frac{y^8}{8!}\right)=\tfrac{1}{2}y^2-\tfrac{1}{24}y^4+\tfrac{1}{720}y^6-\tfrac{1}{40{,}320}y^8$$

$\dfrac{1}{\sqrt{1-y^2}}=\left(1+\left(-y^2\right)\right)^{-\frac{1}{2}}$, so use the binomial series to get: $\left(1+\left(-y^2\right)\right)^{-\frac{1}{2}}\approx$

$$1+\left(-\tfrac{1}{2}\right)\left(-y^2\right)+\frac{\left(-\frac{1}{2}\right)\left(-\frac{1}{2}-1\right)}{2!}\left(-y^2\right)^2+\frac{\left(-\frac{1}{2}\right)\left(-\frac{1}{2}-1\right)\left(-\frac{1}{2}-2\right)}{3!}\left(-y^2\right)^3+\frac{\left(-\frac{1}{2}\right)\left(-\frac{1}{2}-1\right)\left(-\frac{1}{2}-2\right)\left(-\frac{1}{2}-3\right)}{4!}\left(-y^2\right)^2$$

$=1+\tfrac{1}{2}y^2+\tfrac{3}{8}y^4+\tfrac{5}{16}y^6+\tfrac{35}{128}y^8$. So, $\dfrac{1}{\sqrt{1-y^2}}-1\approx\tfrac{1}{2}y^2+\tfrac{3}{8}y^4+\tfrac{5}{16}y^6+\tfrac{35}{128}y^8$.

By examining the polynomial approximations, one can see that they all differ in the 2nd term. Since y is near 0, we can ignore all of the terms beyond the 2nd term.

Thus, $\ln\left(1+\frac{y^2}{2}\right)<1-\cos y<\dfrac{1}{\sqrt{1-y^2}}-1$. (a<b<c)

5. Examine $f(x)=\cos x$ on the interval $[0,1]$ with $a=0$.

$$\cos x=1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\frac{x^8}{8!}-.....$$

Since we only have even powers, assume n is even and write

$$\cos x=x^0-\frac{x^2}{2!}+....+(-1)^{\frac{n}{2}}\frac{x^n}{n!}+E_n(x)$$

$$\left|E_n(x)\right|=\left|f(x)-P_n(x)\right|\le\frac{M}{(n+1)!}|x|^{n+1}$$

where $\max\left|f^{(n+1)}(x)\right|\le M$ on the interval between 0 and x.

Since $\pm\cos x$ and $\pm\sin x$ are both ≤ 1 on the interval $[0,1]$, we can let $M=1$.

Thus, $\left|E_n(x)\right|\le\dfrac{|x|^{n+1}}{(n+1)!}$.

(a) To estimate cos 1, $\left|E_n(1)\right|\le\dfrac{1}{(n+1)!}$. The phrase "accurate to at least 4 decimal places", means that the error is $\le 10^{-4}$. So, set $\dfrac{1}{(n+1)!}\le 10^{-4}$.

This means that we must have $(n+1)!\ge 10^4$.

Since $7!<10^4<8!$, $n+1$ must be at least 8, so $n=7$.

Thus, $\cos 1\approx 1-\dfrac{1}{2!}+\dfrac{1}{4!}-\dfrac{1}{6!}\approx 0.5402777778$.

(b) If we want to be accurate to at least 6 decimal places, then we want to make sure that $\left|E_n(1)\right| \le 10^{-6}$.

Which means that we must have $(n+1)! \ge 10^6$.

Since $9! < 10^6 < 10!$, $n+1$ must be at least 10, so $n = 9$.

Thus, $\cos 1 \approx 1 - \frac{1}{2!} + \frac{1}{4!} - \frac{1}{6!} + \frac{1}{8!} \approx 0.5403025794$.

6. Show $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots + \frac{x^n}{n!} + \ldots\ldots$

1[st]: Show that $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ converges for any real number x.

(In other words, begin with the Taylor polynomial centered at zero and show that its radius of convergence is infinite)

$$\lim_{n\to\infty} \frac{\left|a_{n+1}\right|}{\left|a_n\right|} = \lim_{n\to\infty} \frac{\left|\frac{x^{n+1}}{(n+1)!}\right|}{\left|\frac{x^n}{n!}\right|} = \lim_{n\to\infty} \frac{|x|}{n+1} = 0 \text{ for any real number } x.$$

So the radius of convergence is infinity, so the series converges for all x.

2[nd]: Show that the infinite series converges to e^x.

The error bound tells us how close the finite series is to the function e^x.

So in order for the infinite series to converge to e^x, the error must converge to zero as $n \to \infty$.

$$\left|E_n(x)\right| = \left|e^x - P_n(x)\right| \le \frac{M}{(n+1)!}|x|^{n+1}$$

where $\max\left|f^{(n+1)}(x)\right| \le M$ for all real numbers.

(Note: $f^{(n+1)}$ means the (n+1)[st] derivative of f)

Since $f^{(n)}(x) = e^x$ for any n, we can let $M = e^x$.

Thus, $\left|E_n(x)\right| \le \frac{e^x \cdot |x|^{n+1}}{(n+1)!}$.

Since $\lim_{n\to\infty} \frac{e^x \cdot |x|^{n+1}}{(n+1)!} = e^x \cdot \lim_{n\to\infty} \frac{|x|^{n+1}}{(n+1)!} = e^x \cdot 0 = 0$, (See bottom of page 500)

the error converges to zero. Which means that the series converges to e^x.