Mth 142 Exam III Worksheet Solutions.

1. a) $\{(-.3)^n\}_{n=1}^{\infty}$

$|-.3| < 1 \longrightarrow$ sequence converges to 0.

b) $\left\{\frac{n}{10} + \frac{10}{n}\right\}_{n=1}^{\infty}$

$\frac{10}{n} \to 0$ as $n \to \infty$ while

$\frac{n}{10}$ becomes unbounded.

Sequence diverges.

c). $\{\cos \pi n\}_{n=1}^{\infty}$

$$\cos \pi n = \begin{cases} 1, & n \text{ even} \\ -1, & n \text{ odd} \end{cases}$$

diverges (flips endlessly between -1 and 1).

d) $\left\{ \dfrac{2n + (-1)^n 5}{4n - (-1)^n 3} \right\}_{n=1}^{\infty}$

$$\frac{2n + (-1)^n 5}{4n - (-1)^n 3} = \frac{2 + (-1)^n 5/n}{4 - (-1)^n 3/n} \to \frac{2}{4} = \frac{1}{2}$$

as $n \to \infty$.

e) $\left\{ \dfrac{\sin n}{n} \right\}_{n=1}^{\infty}$

Since $-1 \le \sin x \le 1$ for any x

$$-\frac{1}{n} \le \frac{\sin n}{n} \le \frac{1}{n}$$

$\Rightarrow \dfrac{\sin n}{n} \to 0$ as $n \to \infty$ by squeeze principle.

2. a) $\sum_{k=0}^{10} 7(3)^k$ finite geom series.

$a = 7, \quad x = 3$

$$\text{Sum} = 7 \cdot \frac{1-3^{11}}{1-3}$$

b) $\sum_{k=1}^{5} 2(-4)^{2k}$ finite geom. series

$a = 2(-4)^2 = 32$

$x = (-4)^2 = 16$

$$\text{Sum} = 32 \cdot \frac{1-16^4}{1-16} \quad \left(= 32 \times \frac{65535}{15} \right)$$

c). $\sum_{k=0}^{\infty} z \cdot y^k, \quad |y| < 1$

Infinite geom series whose ratio has absolute value $< 1 \Rightarrow$ convergent

$a = z, \quad x = y.$

$$\text{Sum} = \frac{a}{1-x} = \frac{z}{1-y}.$$

d) $\sum_{k=0}^{\infty} 3(4)^k$

Infinite geom series with radio 4 and $|4| > 1$. $\rightarrow$ Divergent.

3. 1) $$\sum_{k=1}^{\infty} \frac{(-1)^k \sqrt{k^2+1}}{2k+1}$$

$$\frac{(-1)^k \sqrt{k^2+1}}{2k+1} = \frac{(-1)^k \sqrt{k^2+1} / k}{(2k+1)/k}$$

$$= (-1)^k \frac{\sqrt{k^2+1} / \sqrt{k^2}}{(2k+1)/k}$$

$$= (-1)^k \frac{\sqrt{\frac{k^2+1}{k^2}}}{(2k+1)/k}$$

$$= (-1)^k \frac{\sqrt{1+\frac{1}{k^2}}}{2+\frac{1}{k}} \not\to 0 \quad \text{as } k \to \infty.$$

Series diverges by the divergence test.

ii) $$\sum_{k=2}^{\infty} \frac{k^2+1}{k^3-k}$$

When k is large

$$\frac{k^2+1}{k^3-k} \approx \frac{k^2}{k^3} = \frac{1}{k}$$ — suggests divergence.

$$\frac{k^2+1}{k^3-k} > \frac{k^2}{k^3-k} > \frac{k^2}{k^3} = \frac{1}{k} > 0.$$

Since $\sum_{k=2}^{\infty} \frac{1}{k}$ diverges (p series $p \leq 1$),

$\sum_{k=2}^{\infty} \frac{k^2+1}{k^3-k}$ also diverges by the comparison test.

iii) $$\sum_{k=1}^{\infty} \frac{k^2-1}{k^4+2k^2},$$

When n is large

$$\frac{k^2-1}{k^4+2k^2} \approx \frac{k^2}{k^4} = \frac{1}{k^2}$$ - suggests convergence.

When $k \geq 1$

$$0 \leq \frac{k^2-1}{k^4+2\ {}^2} < \frac{k^2}{k^4+2\ {}^2} < \frac{k^2}{k^4} = \frac{1}{k^2}.$$

Since $\sum_{k=1}^{\infty} \frac{1}{k^2}$ converges (p-series $p > 1$),

$\sum_{k=1}^{\infty} \frac{k^2-1}{k^4+2k^2}$ also converges by the comparison test.

N.b. The limit comparison test would also have worked with ii) and iii).

iv) $$\sum_{k=2}^{\infty} \frac{k^2+1}{k^4-k^2-1}$$

When k is large

$$\frac{k^2+1}{k^4-k^2-1} \approx \frac{k^2}{k^4} = \frac{1}{k^2}$$ – suggests convergence.

However, this goes the wrong way for an easy comparison with $\sum \frac{1}{k^2}$.

Instead, try limit comparison with

$$a_k = \frac{k^2+1}{k^4-k^2-1}, \quad b_k = \frac{1}{k^2}$$

$$\frac{a_k}{b_k} = \frac{\frac{k^2+1}{k^4-k^2-1}}{\frac{1}{k^2}} = \frac{k^2(k^2+1)}{k^4-k^2-1}$$

$$= \frac{k^4+k^2}{k^4-k^2-1}$$

$$= \frac{1+\frac{1}{k^2}}{1-\frac{1}{k^2}-\frac{1}{k^4}} \longrightarrow 1 \text{ as } k \to \infty.$$

Since $\sum_{k=2}^{\infty} \frac{1}{k^2}$ converges (p series with $p=2>1$), $\sum_{k=2}^{\infty} \frac{k^2+1}{k^4-k^2-1}$ also converges by the limit comparison test.

v) $\displaystyle\sum_{k=1}^{\infty} \frac{2^{2k}}{3^k (2k)!}$

Try ratio test

Here $\quad a_k = \dfrac{2^{2k}}{3^k (2k)!}$

$$a_{k+1} = \frac{2^{2(k+1)}}{3^{k+1}(2(k+1))!} = \frac{2^{2k+2}}{3^{k+1}(2k+2)!}$$

$$\frac{|a_{k+1}|}{|a_k|} = \frac{\dfrac{2^{2k+2}}{3^{k+1}(2k+2)!}}{\dfrac{2^{2k}}{3^k(2k)!}}$$

$$= \frac{2^{2k+2}}{3^{k+1}(2k+2)!} \cdot \frac{3^k (2k)!}{2^{2k}}$$

$$= \frac{2^{2k+2}}{2^{2k}} \cdot \frac{3^k}{3^{k+1}} \cdot \frac{(2k)!}{(2k+2)!}$$

$$= 2^2 \cdot \frac{1}{3} \cdot \frac{(\cancel{2k})(\cancel{2k-1}) \cdots \cancel{2} \cdot \cancel{1}}{(2k+2)(2k+1)(\cancel{2k})(\cancel{2k-1}) \cdots \cancel{2} \cdot \cancel{1}}$$

$$= 2^2 \cdot \frac{1}{3} \cdot \frac{1}{(2k+2)(2k+1)} \longrightarrow 0 \quad \text{as } k \to \infty.$$

Since $\lim_{k \to \infty} \frac{|a_{k+1}|}{|a_k|}$ exists and is < 1,

the series $\sum_{k=1}^{\infty} \frac{2^{2k}}{3^k (2k)!}$ converges

(absolutely) by the ratio test.

vi) $$\sum_{k=1}^{\infty} \frac{4^k}{2^{k^2}}$$

Try k th root test

$$\sqrt[k]{\frac{4^k}{2^{k^2}}} = \frac{\sqrt[k]{4^k}}{\sqrt[k]{2^{k^2}}} = \frac{(4^k)^{\frac{1}{k}}}{(2^{k^2})^{\frac{1}{k}}} = \frac{4^1}{2^k}$$

$$= \frac{4}{2^k} \longrightarrow 0$$

as $k \to \infty$.

Since $\lim_{k \to \infty} \sqrt[k]{|a_k|}$ exists and is < 1, this series converges by the k th root test.

vii) $\displaystyle\sum_{k=2}^{\infty} \frac{1}{k \ln k}$

Try integral test.

$\frac{1}{x \ln x}$ is a continuous decreasing function on $[2, \infty)$.

Look at $\displaystyle\int_2^{\infty} \frac{1}{x \ln x}\,dx$.

$$\int_2^{b} \frac{1}{x \ln x}\,dx = \int_{\ln 2}^{\ln b} \frac{du}{u}$$

where $u = \ln x$
$du = \frac{dx}{x}$
When $x = 2$, $u = \ln 2$
$x = b$, $u = \ln b$

$$= \Big[\ln |u|\Big]_{\ln 2}^{\ln b}$$

$$= \ln(\ln b) - \ln(\ln 2)$$

$\longrightarrow \infty$ as $b \to \infty$ (very slowly).

Since $\displaystyle\int_2^{\infty} \frac{1}{x \ln x}\,dx$ diverges, the series

$\displaystyle\sum_{k=2}^{\infty} \frac{1}{k \ln k}$ also diverges by the integral test.

viii) $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^k}{\sqrt{k}}$

$|a_k| = \dfrac{1}{\sqrt{k}}$

Then $|a_{k+1}| = \dfrac{1}{\sqrt{k+1}} < \dfrac{1}{\sqrt{k}} = |a_k|$

and $a_k \to 0$ as $k \to \infty$.

Hence this series converges by the Alternating Series Test.

N.b. $\displaystyle\sum_{k=1}^{\infty} |a_k| = \sum_{k=1}^{\infty} \frac{1}{\sqrt{k}} = \sum_{k=1}^{\infty} \frac{1}{k^{1/2}}$

which is a divergent p series ($p = \frac{1}{2} \leq 1$).

Hence this series is only conditionally convergent.