Mth 142 Worksheet # 7 Solutions

1. a) $y' = \frac{1}{x}$

slope becomes vertical on the y-axis ($x=0$), negative for $x<0$ and positive for $x>0$.

Only <u>IV</u> fits these criteria.

b) $y' = \frac{1}{y}$

Slope becomes vertical on the x-axis ($y=0$), negative for $y<0$ and positive for $y>0$

Only <u>VI</u> fits these criteria

c) $y' = e^{-x^2}$

Slope is always positive and non-zero and tends rapidly to 0 as $x \to \pm\infty$.

Only $\underline{\text{V}}$ fits these criteria

d) $y' = y^2 - 1$

Slope is 0 when $y = \pm 1$, > 0 when $y < -1$ or $y > 1$ and < 0 when $-1 < y < 1$.

Only $\underline{\text{II}}$ fits these criteria.

e) $y' = \dfrac{x+y}{x-y}$

Slope is 0 when $y = -x$ and ∞ (vertical) when $y = x$.

Only I fits these criteria.

f) $y' = (\sin x)(\sin y)$

Slope is 0 whenever x or y is an integer multiple of π. In particular, slope is 0 on both the x- and y-axes.

Only III fits these criteria.

Summary

a) $\longleftrightarrow$ IV	d) $\longleftrightarrow$ II
b) $\longleftrightarrow$ VI	e) $\longleftrightarrow$ I
c) $\longleftrightarrow$ V	f) $\longleftrightarrow$ III

2\. a) $\left\{ \dfrac{n^3}{(n+1)(3n^2+7)} \right\}_{n=1}^{\infty}$

Top power of n in both top and bottom is n^3, so divide above and below by n^3

$$\frac{n^3}{(n+1)(3n^2+7)} = \frac{n^3/n^3}{(n+1)/n \; (3n^2+7)/n^2}$$

$$= \frac{1}{(1+\frac{1}{n})(3+7/n^2)}$$

$$\to \frac{1}{(1)(3)} = \frac{1}{3} \quad \text{as } n \to \infty$$

Convergent !.

b) $\left\{ (-1)^n \dfrac{2n^3}{n^3+1} \right\}_{n=1}^{\infty}$

Again we divide above and below by n^3

$$(-1)^n \frac{2n^3}{n^3+1} = (-1)^n \frac{2}{1+\frac{1}{n^3}}$$

$$\frac{2}{1+\frac{1}{n^3}} \longrightarrow 2 \quad \text{as} \quad n \to \infty, \text{ so}$$

the long-term behavior of

$$(-1)^n \frac{2n^3}{n^3+1}$$

is to flip indefinitely between -2 and 2.

Diverges.

c). $$\left\{ \frac{\sqrt{n^4+1}}{n^2+17n} \right\}_{n=1}^{\infty}$$

$\sqrt{n^4+1} \approx \sqrt{n^4} = n^2$ for n large

so we divide above and below by n^2

$$\frac{\sqrt{n^4+1}}{n^2+17n} = \frac{\sqrt{n^4+1}/n^2}{(n^2+17n)/n^2}$$

$$= \frac{\sqrt{\frac{n^4+1}{n^4}}}{1+\frac{17}{n}}$$

[n^2 becomes n^4 once we take it inside the square root

$$= \frac{\sqrt{1+\frac{1}{n^4}}}{1+\frac{17}{n}} \longrightarrow \frac{\sqrt{1+0}}{1+0} = 1$$

as $n \to \infty$

Convergent.

3. a) $\frac{1}{2}, \frac{3}{4}, \frac{5}{6}, \frac{7}{8}, \cdots$

Both top and bottom increase by 2 as n increases by 1 and so are of the form $2n+k$.

Checking the first term $\frac{1}{2}$ gives $k=-1$ for the top and $k=0$ for the second. The general formula is thus

$$\left\{ \frac{2n-1}{2n} \right\}_{n=1}^{\infty}.$$

b) $\frac{1}{3^5}, -\frac{1}{3^8}, \frac{1}{3^{11}}, -\frac{1}{3^{14}}$

Top flips up and down between 1 and -1 and starts at 1

. $(-1)^{n-1}$ works

Bottom is a power of 3 which starts at 5 and grows by 3 with each term. Power is thus of the form $3n+k$ and checking the first term gives $k = 2$.

Putting all this together gives a general formula

$$\left\{ \frac{(-1)^{n-1}}{3^{3n+2}} \right\}_{n=1}^{\infty}.$$

4. a) $\sum_{k=1}^{\infty} \left(\frac{2}{3}\right)^{k+2}$

Geom. series with ratio $r = \frac{2}{3}$ and $\left|\frac{2}{3}\right| < 1$ so this series converges.

The first term is $a = \left(\frac{2}{3}\right)^3 = \frac{8}{27}$

and so the sum is

$$\frac{a}{1-r} = \frac{8/27}{1-2/3} = \frac{\frac{8}{27}}{1/3} = \frac{8}{9}.$$

b) $\sum_{k=1}^{\infty} \left(-\frac{3}{2}\right)^{k+1}$

Geom. series with ratio $r = -\frac{3}{2}$ and $\left|\frac{-3}{2}\right| > 1$, so this series diverges.

c) $\displaystyle\sum_{k=1}^{\infty} \frac{1}{9k^2+3k-2}$

Factor the denominator

$$\frac{1}{9k^2+3k-2} = \frac{1}{(3k+2)(3k-1)}$$

Now try for partial fractions

$$\frac{1}{(3k+2)(3k-1)} = \frac{A}{3k+2} + \frac{B}{3k-1}$$

$$= \frac{A(3k-1) + B(3k+2)}{(3k+2)(3k-1)}$$

$$= \frac{k(3A+3B) - A + 2B}{(3k+2)(3k-1)}.$$

Equating powers of k in the numerators

$k^1/ \quad 3A + 3B = 0 \quad (1)$

$k^0/ \quad -A + 2B = 1 \quad (2).$

(1) $\Rightarrow$ $B = -A$. Now subst for B in (2) to get

$$-A - 2A = 1$$

$$A = -\frac{1}{3}$$

$$B = -A = +\frac{1}{3}.$$

So

$$\frac{1}{(3k+2)(3k-1)} = \frac{-1}{3(3k+2)} + \frac{1}{3(3k-1)}$$

$$= \frac{1}{3}\left(\frac{1}{3k-1} - \frac{1}{3k+2}\right)$$

Now look at the nth partial sum S_n.

$$S_n = \frac{1}{3}\left(\frac{1}{2} - \frac{1}{5}\right)$$
$$+ \frac{1}{3}\left(\frac{1}{5} - \frac{1}{8}\right)$$
$$+ \frac{1}{3}\left(\frac{1}{8} - \frac{1}{11}\right)$$
$$+ \vdots$$
$$+ \frac{1}{3}\left(\frac{1}{3n-1} - \frac{1}{3n+2}\right)$$

This is a telescoping sum where nearly all the terms cancel and we're left with

$$S_n = \frac{1}{3} \cdot \frac{1}{2} - \frac{1}{3} \cdot \frac{1}{3n+2}$$

$$\to \frac{1}{3} \cdot \frac{1}{2} = \frac{1}{6} \quad \text{as } n \to \infty.$$

Hence the sequence of partial sums converges to $\frac{1}{6}$ and so the series

$$\sum_{k=1}^{\infty} \frac{1}{9k^2 + 3k - 2}$$

is convergent with sum $\frac{1}{6}$.