Mth 142 0005

Exam II Review Worksheet

Solutions.

1. $f(x) = 2x - x^2, \quad g(x) = x^2$

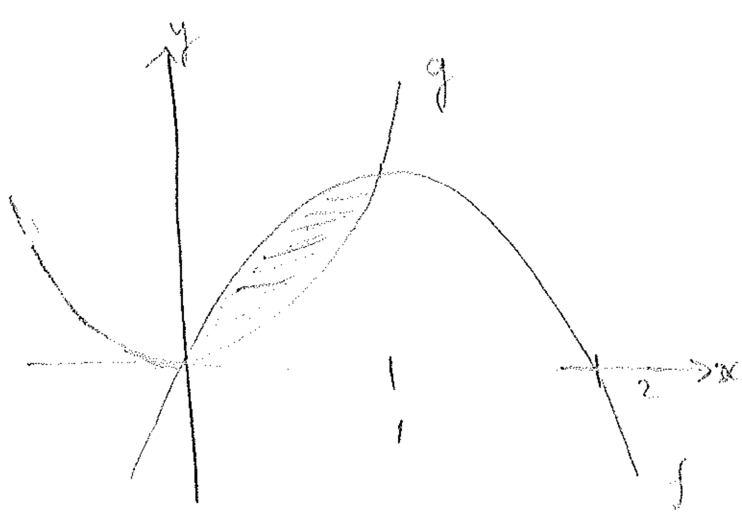
Equate the two functions to find the limits of integration

$2x - x^2 = x^2$

$2x - 2x^2 = 0$

$2x(1-x) = 0 \quad \rightarrow \quad x = 0, 1$

So $a = 0$, $b = 1$



Which function is larger?

Test at $\frac{1}{2}$ $\left(0 < \frac{1}{2} < 1\right)$

$$f\left(\tfrac{1}{2}\right) = 2\left(\tfrac{1}{2}\right) - \left(\tfrac{1}{2}\right)^2 = 3/4$$

$$g\left(\tfrac{1}{2}\right) = \left(\tfrac{1}{2}\right)^2 = \frac{1}{4}.$$

Hence the area between the two curves is given by

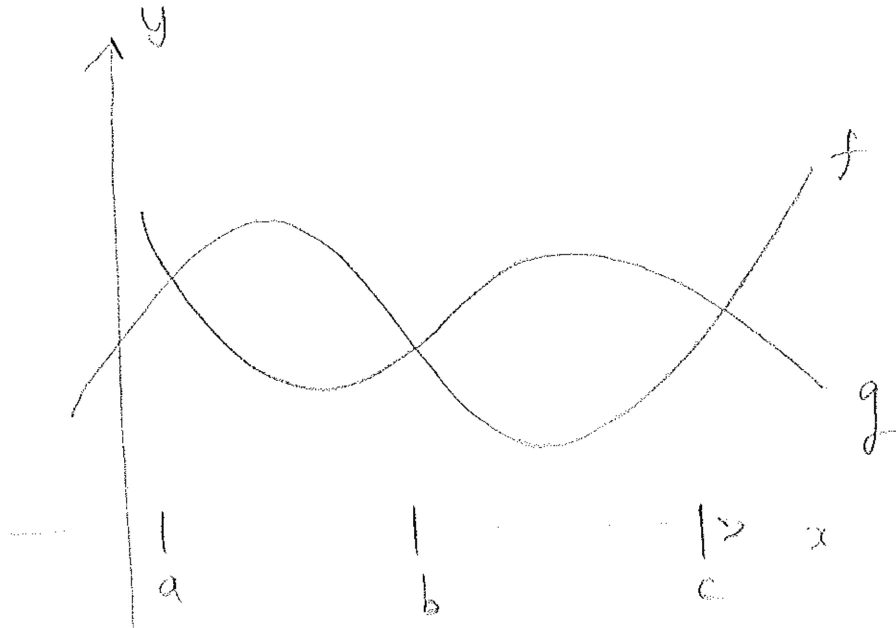
$$A = \int_a^b (f(x) - g(x))\,dx$$

$$= \int_0^1 (2x - x^2 - x^2)\,dx$$

$$= \int_0^1 (2x - 2x^2)\,dx$$

$$= \left[x^2 - \frac{2x^3}{3}\right]_0^1 = 1 - \frac{2}{3} - 0 = \frac{1}{3}.$$

2.



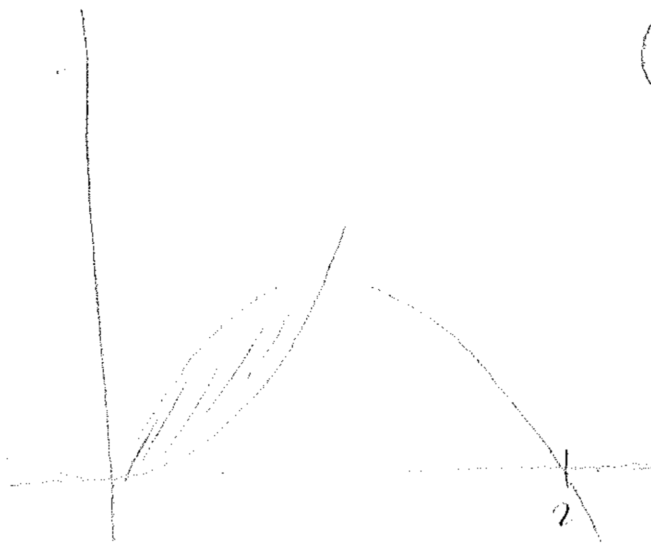
On $[a,b]$ $f(x) \geq g(x)$ and
on $[b,c]$ $g(x) \geq f(x)$.

Hence the area enclosed between the two graphs is given by

$$A = \int_a^b (f(x) - g(x))\,dx + \int_b^c (g(x) - f(x))\,dx.$$

3.

$\circlearrowleft y = 2$



The distances to the axis of rotation are given by

$$2 - f(x) = 2 - 2x + x^2,$$
$$2 - g(x) = 2 - x^2.$$

Which curve is furthest from the axis of rotation? Test at $\frac{1}{2}$ again

$$2 - f(\tfrac{1}{2}) = 2 - 2(\tfrac{1}{2}) + (\tfrac{1}{2})^2 = \tfrac{5}{4}$$
$$2 - g(\tfrac{1}{2}) = 2 - (\tfrac{1}{2})^2 = \tfrac{7}{4}.$$

Hence the inner radius is given by $2 - f(x)$ and the outer radius by $2 - g(x)$.

Then

$$V = \pi \int_a^b (\text{outer}^2 - \text{inner}^2)\,dx$$

$$= \pi \int_0^1 \left((2 - x^2)^2 - (2 - 2x + x^2)^2 \right) dx$$

$$= \pi \int_0^1 \Big(\cancel{4} - 4x^2 + \cancel{x^4} - (\cancel{4} + 4x^2 + \cancel{x^4}$$
$$- 8x + 4x^2 - 4x^3) \Big) dx$$

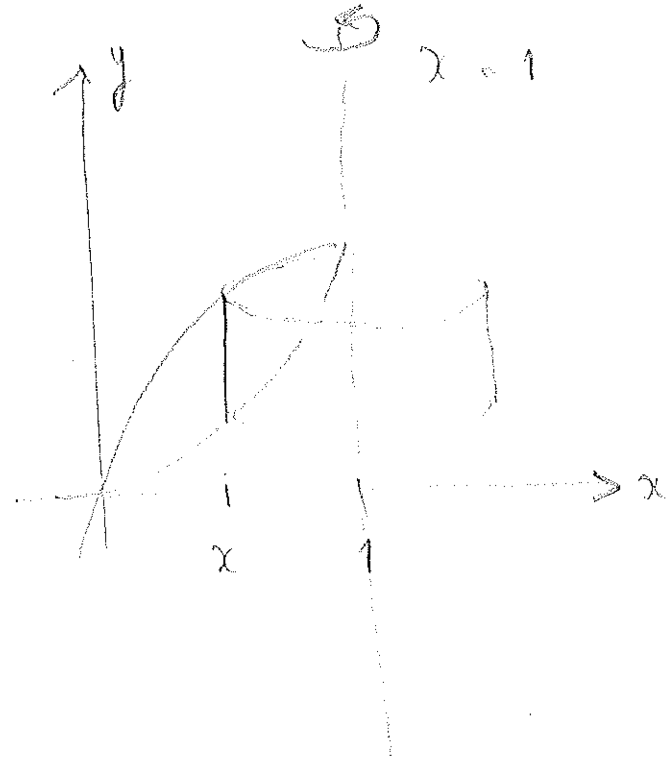
$$= \pi \int_0^1 (8x - 12x^2 + 4x^3)\,dx$$

$$= \pi \Big[4x^2 - 4x^3 + x^4 \Big]_0^1$$

$$= \pi (4 - 4 + 1 - 0)$$

$$= \pi.$$

4.



For $0 \le x \le 1$, the distance to the axis of rotation at $x = 1$ is given by

$$r = 1 - x$$

while the height of the cylinder is given by

$$f(x) - g(x) = 2x - 2x^2$$

The volume of revolution is then

$$V = 2\pi \int_a^b \text{radius} \times \text{height} \; dx$$

$$= 2\pi \int_0^1 (2x - 2x^2)(1 - x)\,dx$$

$$= 2\pi \int_0^1 (2x - 2x^2 - 2x^2 + 2x^3)\,dx$$

$$= 2\pi \int_0^1 (2x - 4x^2 + 2x^3)\,dx$$

$$= 2\pi \left[x^2 - \frac{4x^3}{3} + \frac{x^4}{2} \right]_0^1$$

$$= 2\pi \left(1 - 4/3 + \frac{1}{2} - 0 \right)$$

$$= 2\pi \cdot \left(-\frac{1}{3} + \frac{1}{2} \right)$$

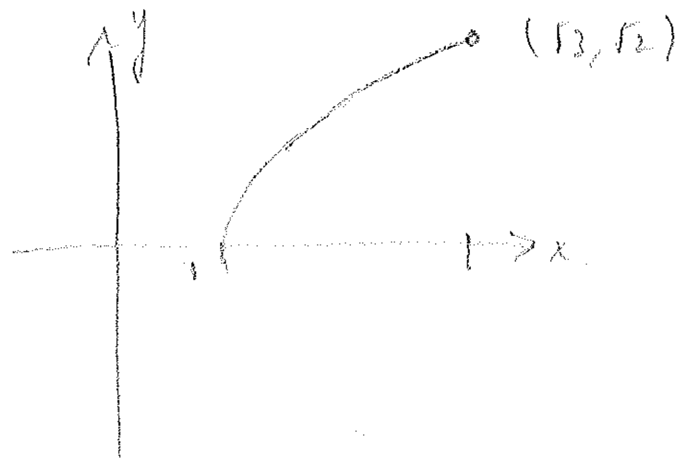
$$= 2\pi \left(\frac{1}{6} \right)$$

$$= \frac{\pi}{3}.$$

5. $x^2 - y^2 = 1$

$$y^2 = x^2 - 1$$

$$y = \sqrt{x^2 - 1}$$



Take positive square root for upper branch of hyperbola.

$$\frac{dy}{dx} = \frac{2x}{2\sqrt{x^2-1}} = \frac{x}{\sqrt{x^2-1}}$$

$$\sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \sqrt{1 + \frac{x^2}{(x^2-1)}} = \sqrt{\frac{x^2 - 1 + x^2}{x^2-1}}$$

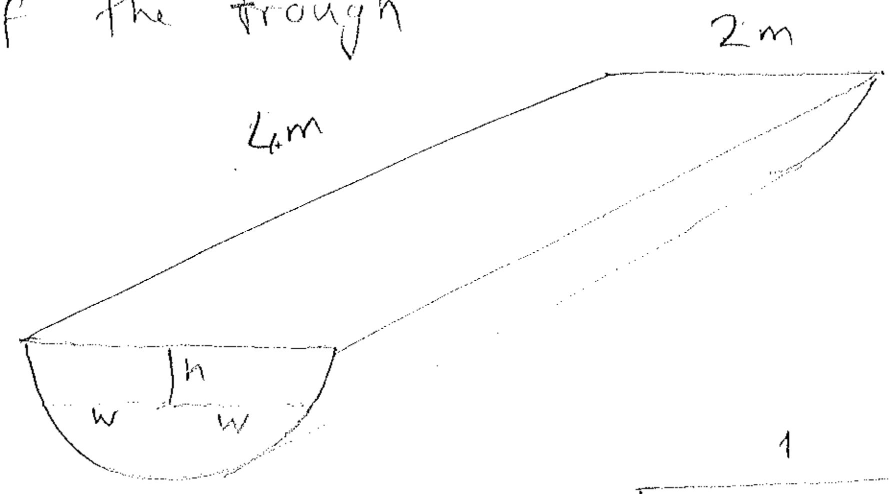
$$= \sqrt{\frac{2x^2-1}{x^2-1}}$$

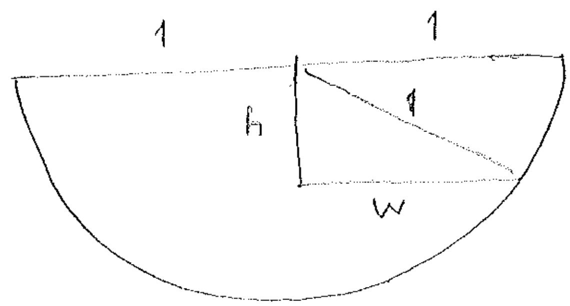
Then

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$= \int_1^{\sqrt{3}} \sqrt{\frac{2x^2-1}{x^2-1}}\, dx$$

6. Let h be the distance from the top of the trough





By Pythagoras

$$h^2 + w^2 = 1$$

$$w = \sqrt{1-h^2}$$

$$\text{Area} = 2w \times 4 = 8w = 8\sqrt{1-h^2}$$

$$\text{Volume} = \text{Area}\, \Delta h = 8\sqrt{1-h^2}\, \Delta h$$

$$\text{Mass} = 1000 \times \text{Volume} = 8000\sqrt{1-h^2}\, \Delta h$$

$$\text{Weight} = 9.81 \times \text{Mass} = 9.81 \times 8000\sqrt{1-h^2}$$

Slice of water has to be raised a distance h metres to reach the top of the trough, so the work done is

$$\Delta W = \text{weight} \times \text{distance}$$

$$= 9.81 \times 8000 \sqrt{1-h^2}\, \Delta h \cdot h$$

$$= 9.81 \times 8000 \sqrt{1-h^2} \cdot h \Delta h.$$

The total work is then approx.

$$W \approx \sum 9.81 \times 8000 \sqrt{1-h^2}\, h \Delta h.$$

and on letting $\Delta h \to 0$, we get.

$$W = 9.81 \times 8000 \int_0^1 \sqrt{1-h^2} \cdot h \, dh$$

Do this by a regular substitution
(<u>NOT</u> a trig substitution)

$$u = 1-h^2, \qquad du = -2h\,dh$$
$$-\frac{du}{2} = h\,dh$$

When $h=0$, $u=1$,
$h=1$, $u=0$

$$= 9.81 \times 8000 \int_1^0 \sqrt{u} \cdot -\frac{du}{2}$$

$$= 9.81 \times 4000 \int_0^1 u^{\frac{1}{2}}\, du \qquad \left(8000 \times \tfrac{1}{2} = 4000 \right)$$

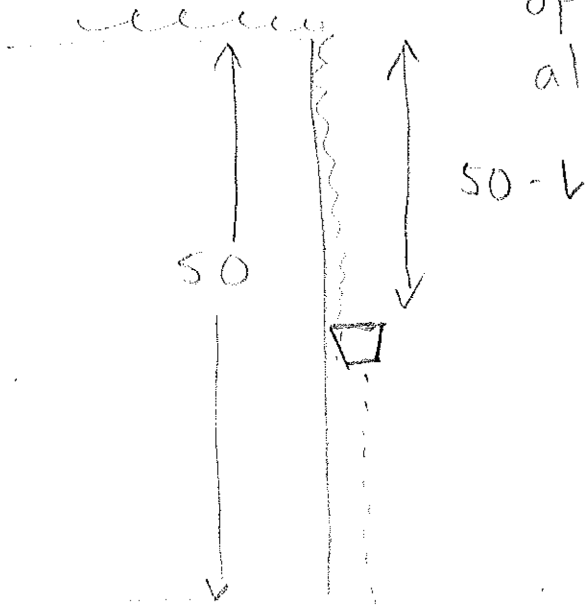
$$= 9.81 \times 4000 \left[\frac{2}{3} u^{3/2} \right]_0^1$$

$$= 9.81 \times 4000 \times \left(\frac{2}{3} - 0 \right)$$

$$= 26,160 \text{ J}$$

7.

Let y be the number of feet of chain already hauled up



Weight of bucket + water as a fn of t

$= 10 + 100 - t$

$= 110 - t$ pounds.

Chain is hauled up at 2ft/sec so

$y = 2t$

$t = y/2$

Hence the weight of the bucket as a function of y is

$110 - y/2$ pounds.

When l ft of chain have been hauled up $50-l$ ft are left hanging over the building so the weight of this portion of chain is

$$2(50-l) = 100-2l \text{ pounds.}$$

The total weight is then

Weight of chain + Weight of bucket

$$= 100-2l + 110 - l/2$$

$$= 210 - 5l/2 \text{ lbs.}$$

The total work is then

$$W = \int \text{force } d(\text{distance})$$

$$= \int_0^{50} \left(210 - \frac{5t}{2}\right) dt$$

$$= \left[210t - \frac{5t^2}{4} \right]_0^{50}$$

$$= 10,500 - \frac{5 \times 2500}{4} - 0$$

$$= 10,500 - 3,125$$

$$= 7,375 \text{ ft lbs.}$$

8. $y = e^{-x^2}$

$y' = -2xe^{-x^2}$ (chain rule)

$y'' = -2e^{-x^2} + 4x^2 e^{-x^2}$ (chain and product rules)

$y'' + 2xy' + 2y$

$= -2e^{-x^2} + 4x^2 e^{-x^2}$

$+ 2x(-2xe^{-x^2})$

$+ 2e^{-x^2}$

$= -2e^{-x^2} + 4x^2 e^{-x^2} - 4x^2 e^{-x^2} + 2e^{-x^2}$

$= 0.$ ✓

$y(0) = e^{0^2} = 1$ ✓

$y'(0) = -2(0)e^{0^2} = 0.$ ✓

9. i) $y' - (1+x)(1+y^2) = 0, \; y(0) = 0$

$$\frac{dy}{dx} = (1+x)(1+y^2)$$

Separate vaiables

$$\frac{dy}{1+y^2} = (1+x)\,dx$$

Integrate both sides

$$\int \frac{dy}{1+y^2} = \int (1+x)\,dx$$

$$\arctan y = x + \frac{x^2}{2} + C$$

Use the IC $y(0) = 0$ to find C

$$\arctan 0 = 0 + \frac{0^2}{2} + C$$

$0 = C.$

So the soln of the IVP is

$$\arctan y = x + x^2/2$$

In this case we can write y explicitly as a fn of x by taking tan of both sides

$$\tan(\arctan y) = \tan(x + x^2/2)$$

$$y = \tan(x + x^2/2).$$

ii) $y' - xe^y = 2e^y, \quad y(0) = 0$

$$\frac{dy}{dx} = xe^y + 2e^y$$

$$\frac{dy}{dx} = (x+2)e^y$$

Seperate variables

$$\frac{dy}{e^y} = (x+2)dx$$

$$e^{-y}dy = (x+2)dx$$

Integrate both sides

$$\int e^{-y} dy = \int (x+2)dx$$

$$-e^{-y} = (x+2) + C$$

Use $y(0) = 0$ to find C

$$-e^{0} = 0 + 2 + C$$

$$-1 = 2 + C$$

$$C = -3.$$

So

$$-e^{-y} = x + 2 - 3$$

which we can tidy up and take ln of both sides to get y in terms of x.

$$-e^{-y} = x - 1$$

$$e^{-y} = 1 - x$$

$$-y = \ln(1 - x)$$

$$y = -\ln(1 - x).$$

iii) $y' = \dfrac{3x^2}{2y + \cos y}$, $y(0) = \pi$.

Separate

$$\frac{dy}{dx} = \frac{3x^2}{2y + \cos y}$$

$$(2y + \cos y)\,dy = 3x^2\,dx$$

Integrate both sides

$$\int (2y + \cos y)\,dy = \int 3x^2\,dx$$

$$y^2 + \sin y = x^3 + C$$

Use $y(0) = \pi$ to find C

$$\pi^2 + \sin \pi = 0^3 + C$$

$\pi^2 = C.$

Hence our solution is

$$y^2 + \sin y = x^3 + \pi^2.$$

In this case we cannot (easily) get y as an explicit function of x.

10. Slope of tangent line at (x, y) is xe^{-y} means

$$\frac{dy}{dx} = xe^{-y}$$

while x intercept is 2 means $y(2) = 0$

Hence we have the IVP

$$\frac{dy}{dx} = xe^{-y}, \qquad y(2) = 0.$$

Separate variables

$$\frac{dy}{e^{-y}} = x\,dx$$

$$e^{y}\,dy = x\,dx$$

Integrate both sides

$$\int e^y \, dy = \int x \, dx$$

$$e^y = \frac{x^2}{2} + C$$

Use the IC $y(2) = 0$ to find C.

$$e^0 = \frac{2^2}{2} + C$$

$$1 = 2 + C$$

$$C = -1$$

So $e^y = \frac{x^2}{2} - 1$

and thus

$$y = \ln\left(\frac{x^2}{2} - 1\right)$$

is the desired soln.

11. 1) Radioactive sample with half-life of 50 yrs and initial quantity of 2 kg.

If we let $Q = Q(t)$ be the amount at time t, then we have the IVP

$$\frac{dQ}{dt} = -kQ, \qquad Q(0) = 2$$

Separate variables

$$\frac{dQ}{Q} = -k\,dt$$

Integrate both sides

$$\int \frac{dQ}{Q} = \int -k\,dt$$

$$\ln|Q| = -kt + C$$

Take exp of both sides

$$|Q| = e^{-kt+c} = e^c \cdot e^{-kt}$$

$$Q = \pm e^c e^{-kt}$$

$$Q = A e^{-kt} \quad \text{where } A = \pm e^c.$$

Use $Q(0) = 2$ to find A.

$$2 = A e^0 = A$$

So

$$Q(t) = 2e^{-kt}.$$

ii) Use the half-life to find k. After 50 yrs we have $\frac{2}{2} = 1$ kg left. So

$$1 = 2e^{-k(50)}$$

$$\frac{1}{2} = e^{-50k}$$

Take ln of both sides

$$\ln\left(\tfrac{1}{2}\right) = \ln\left(e^{-50k}\right)$$

$$= -50k$$

So $k = \ln\left(\tfrac{1}{2}\right)\Big/{-50} = \dfrac{\ln 2}{50}$

$$\approx .01386$$

and

$$Q = 2e^{-.01386t} \quad kg.$$

iii) When $t = 75$ yrs, we have

$$2e^{-.01386 \times 75} \approx .707 \ kg$$

$$\left(= \tfrac{1}{\sqrt{2}} \, kg \ \ (\text{why?})\right)$$