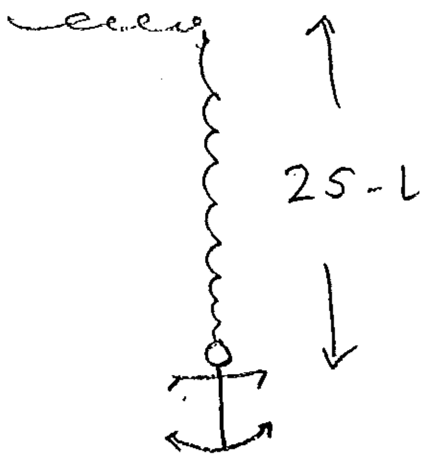
Worksheet #5 Solutions

1.



Let L be the length in feet of the chain already pulled up.

Then there is still $25 - L$ ft of chain in the water and the combined weight of this amount of chain and the anchor is

$$(25 - L)(3) + 100$$

$$= 175 - 3L \text{ lbs.}$$

To raise this portion of the chain and the anchor a small distance ΔL, the work done is

$$\text{Weight} \times \Delta L = (175 - 3L)\,\Delta L$$

Adding contributions gives

$$W \approx \sum (175 - 3L)\,\Delta L.$$

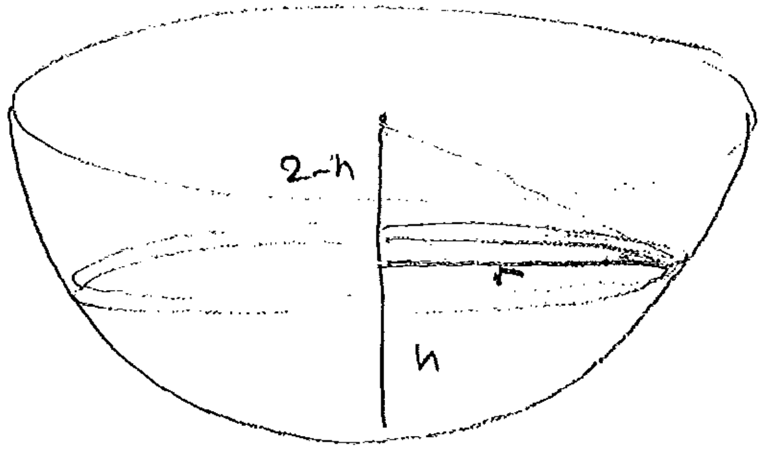
If we then let $\Delta L \to 0$ and remember that L runs from 0 ft (no chain pulled up) to 25 ft (all chain pulled up), then

$$W = \int_0^{25} (175 - 3L)\,dL$$

$$= \left[175L - \frac{3L^2}{2}\right]_0^{25}$$

$$= 175(25) - \frac{3(25)^2}{2} = 3437.5 \text{ ft lbs.}$$

2.

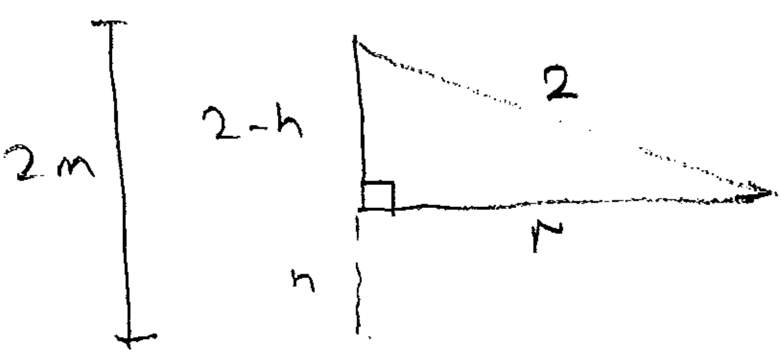


Let h be the depth (in m) of the water in the tank.

Surface is a circle of radius r and the volume of a thin slice is then

$$\Delta V = \pi r^2 \Delta h.$$

To get r^2 in terms of h, use Pythagoras.



$$r^2 + (2-h)^2 = 2^2$$

$$r^2 = 2^2 - (2-h)^2$$

$$= 4h - h^2.$$

Then, substituting for r

$$\Delta V = \pi (4h - h^2) \Delta h$$

The density of water is $1000\ kg/m^3$ and $g = 9.81\ m/s^2$, so the weight is

$$9.81 \times 1000 \times \pi (4h - h^2) \Delta h.$$

We need to raise this slice a distance h from the bottom of the tank.

The work done is then

$$\Delta W \approx \text{weight} \times h = 9.81 \times 1000 \times \pi (4h - h^2) \Delta h \cdot h$$

$$= 9.81 \times 1000 \pi (4h^2 - h^3) \Delta h.$$

Add the contributions together

$$W \approx \sum 9.81 \times 1000 \pi (4h^2 - h^3) \Delta h.$$

Now let $\Delta h \to 0$ and remember that h runs from $0\,m$ (empty) to $2\,m$ (full) to get

$$W = \int_0^2 9.81 \times 1000\pi (4h^2 - h^3)\, dh$$

$$= 9.81 \times 1000\pi \int_0^2 (4h^2 - h^3)\, dh$$

$$= 9.81 \times 1000\pi \left[\frac{4h^3}{3} - \frac{h^4}{4}\right]_0^2$$

$$= 9.81 \times 1000\pi \left(\frac{32}{3} - \frac{16}{4} - 0\right)$$

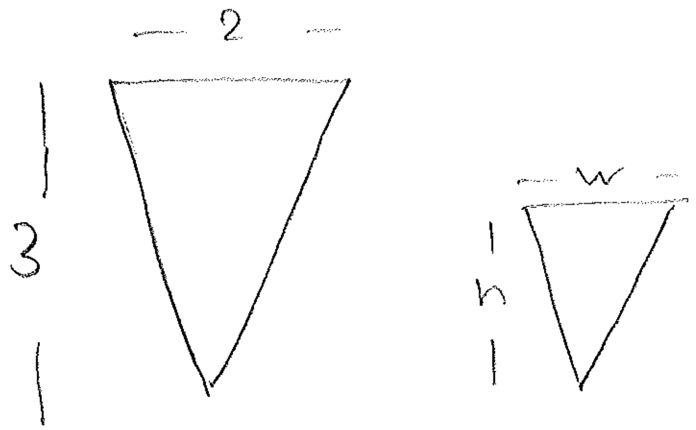
$$= 9.81 \times 1000\,\pi \cdot \frac{20}{3} \text{ Joules}$$

$$\approx 205{,}460 \text{ Joules.}$$

3. Let h be the depth of a slice of water in the trough.

Let w be the width of this slice (the length is constant at 15 ft).

Find w in terms of h using similar triangles



$$\frac{w}{h} = \frac{2}{3}$$

$$w = \frac{2h}{3}.$$

Slice has area $15w = 15\left(\frac{2h}{3}\right) = 10h$

and volume $10h\,\Delta h$.

The weight is then $(62.4)(10h)\Delta h$
$= 624h\,\Delta h$.

This slice needs to be raised a distance of 3 - h feet to clear the top of the tank.

The work required for this is

$$\Delta W \approx \text{Force} \times \text{Dist}$$

$$= (624\, h\, \Delta h)(3-h)$$

$$= 624\, h(3-h)\, \Delta h$$

$$= 624\,(3h - h^2)\, \Delta h.$$

The total work is then

$$W \approx \sum 624(3h - h^2)\, \Delta h$$

and as we let $\Delta h \to 0$

$$W = \int_0^3 624\,(3h - h^2)\, dh$$

$$= 624 \int_0^3 (3h - h^2)\,dh$$

$$= 624 \left[\frac{3h^2}{2} - \frac{h^3}{3} \right]_0^3$$

$$= 624 \left(\frac{27}{2} - \frac{27}{3} - 0 \right)$$

$$= 624 \left(\frac{27}{6} \right)$$

$$= 624 \left(\frac{9}{2} \right)$$

$$= 328 \times 9$$

$$= 2952 \text{ ft lbs.}$$

4. $f(x) = \cos x$

$f'(x) = -\sin x$

$$\sqrt{1+(f'(x))^2} = \sqrt{1+(-\sin x)^2}$$

$$= \sqrt{1+\sin^2 x}.$$

$$L = \int_a^b \sqrt{1+(f'(x))^2}\,dx$$

$$= \int_0^{\frac{\pi}{4}} \sqrt{1+\sin^2 x}\,dx.$$

5. $x^2 + y^2 = 1$

$$y = +\sqrt{1-x^2} \quad \text{for upper semicircle.}$$

$$\frac{dy}{dx} = \frac{-2x}{2\sqrt{1-x^2}} = \frac{-x}{\sqrt{1-x^2}}$$

So $$\sqrt{1+\left(\frac{dy}{dx}\right)^2} = \sqrt{1+\left(\frac{-x}{\sqrt{1-x^2}}\right)^2}$$

$$= \sqrt{1+\frac{x^2}{1-x^2}}$$

$$= \sqrt{\frac{1-x^2+x^2}{1-x^2}}$$

$$= \frac{1}{\sqrt{1-x^2}}.$$

Then

$$L = \int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$$

$$= \int_{-1}^{1} \frac{1}{\sqrt{1-x^2}}\,dx$$

$$= \Big[\arcsin x\Big]_{-1}^{1}$$

[can be done either as a known antid. or using trig substitution]

$$= \arcsin 1 - \arcsin(-1)$$

$$= \frac{\pi}{2} - \left(-\frac{\pi}{2}\right)$$

$$= \pi$$

This is as we'd expect since the length of a semicircle is

$\frac{1}{2}\, 2\pi \times$ radius $= \pi \times$ radius

which in our case is

$\pi \times 1 = \pi$.

6. $y = 2e^{2x}$

$y' = 2(2)e^{2x} = 4e^{2x}$

So $y' - 2y = 4e^{2x} - 2(2e^{2x})$

$= 0$

and y satisfies the DE.

Also $y(0) = 2e^{0} = 2$ and

thus y also satisfies the IC.

7. $y = e^x + \sin(2x)$

$y' = e^x + 2\cos(2x)$

$y'' = e^x - 4\sin(2x)$.

So $y'' + 4y = e^x - 4\sin 2x$
$+ 4(e^x + \sin 2x)$

$= 5e^x$

and y is a soln of the DE.

Also $y(0) = e^0 + \sin(0) = 1$

$y'(0) = e^0 + 2\cos(0) = 1 + 2 = 3$

So y also satisfies the ICs.