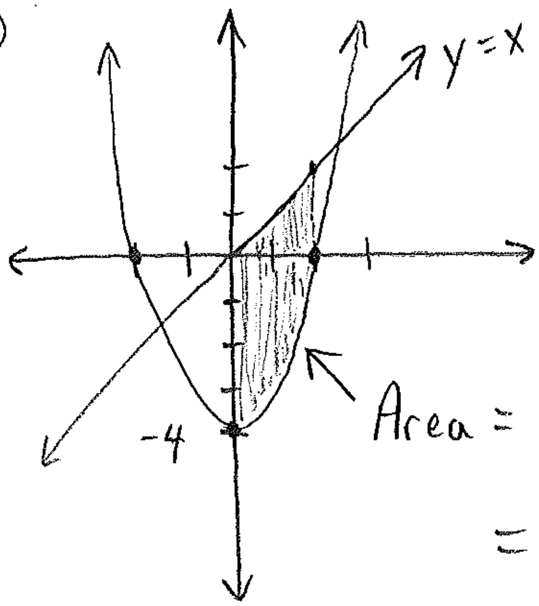
MTH 142 Worksheet #4 SOLUTIONS

#1



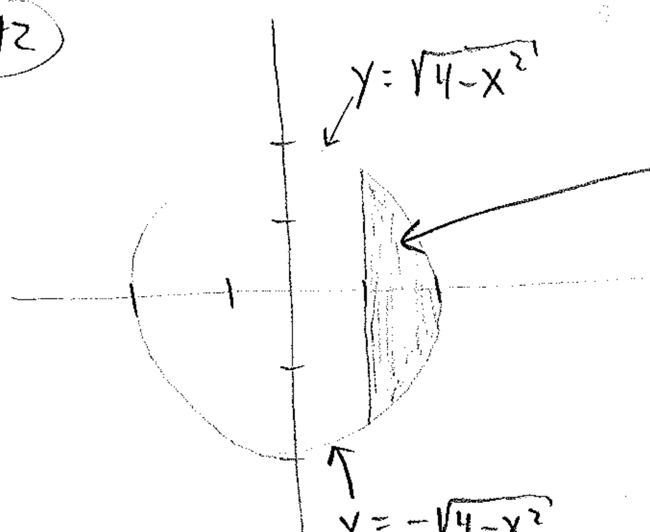
$$\text{Area} = \int_0^2 (x-(x^2-4))\,dx$$

$$= \int_0^2 (-x^2+x+4)\,dx$$

$$= -\tfrac{1}{3}x^3+\tfrac{1}{2}x^2+4x\Big|_0^2$$

$$= -\tfrac{8}{3}+\tfrac{4}{2}+8 = 7\tfrac{1}{3}$$

#2



$$A = 2\cdot\int_1^2 \sqrt{4-x^2}\,dx$$

Let $x = 2\cdot\sin\theta$

$dx = 2\cdot\cos\theta\,d\theta$

So, $\int \sqrt{4-x^2}\,dx$

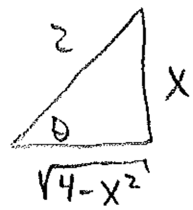
$$= 2\int \sqrt{4-4\sin^2\theta}\,\cos\theta\,d\theta$$

$$= 4\int \cos^2\theta\,d\theta \quad \left(\text{note: } \tfrac{\pi}{6}\le\theta\le\tfrac{\pi}{2}\right)$$

$$= 4\cdot\left(\tfrac{1}{2}\cos\theta\cdot\sin\theta + \tfrac{1}{2}\int 1\,d\theta\right)$$

$= 4\left(\frac{1}{2}\cos\theta\cdot\sin\theta + \frac{\theta}{2}\right) = 2\cos\theta\sin\theta + 2\theta$

$\sin\theta = \frac{x}{2}$



So, $\cos\theta = \frac{\sqrt{4-x^2}}{2}$

$\theta = \sin^{-1}\left(\frac{x}{2}\right)$

So, $\int\sqrt{4-x^2}\,dx = 2\cdot\frac{\sqrt{4-x^2}}{2}\cdot\frac{x}{2} + 2\cdot\sin^{-1}\left(\frac{x}{2}\right) + C$

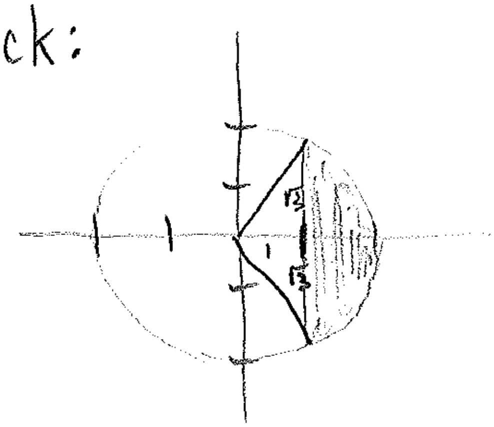
Thus, $A = 2\cdot\int_1^2\sqrt{4-x^2}\,dx$

$$= 2\cdot\left(\frac{x\sqrt{4-x^2}}{2} + 2\cdot\sin^{-1}\left(\frac{x}{2}\right)\Big|_1^2\right)$$

$$= 2\cdot\left(\pi - \left(\frac{\sqrt{3}}{2} + \frac{\pi}{3}\right)\right)$$

$$= 2\cdot\left(\frac{2\pi}{3} - \frac{\sqrt{3}}{2}\right) = \frac{4\pi}{3} - \sqrt{3}$$

Check:

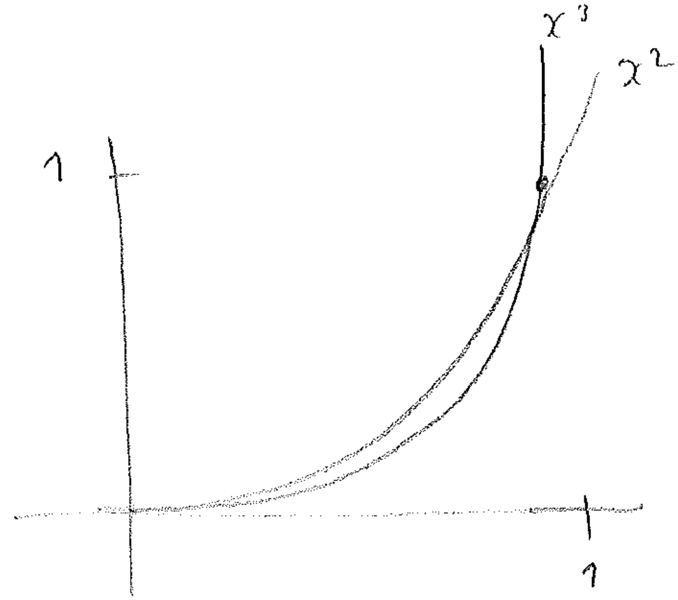


Area = sector  − triangle 

$= \frac{1}{2}\cdot 2^2\cdot\frac{2\pi}{3} - \sqrt{3}$

$= \frac{4\pi}{3} - \sqrt{3}$ ✓

3.



Intersect the two curves to find the limits of integration:

$$x^2 = x^3$$

$$x^2 - x^3 = 0$$

$$x^2(x-1) = 0$$

$$x = 0, 1$$

So $a=0$, $b=1$.

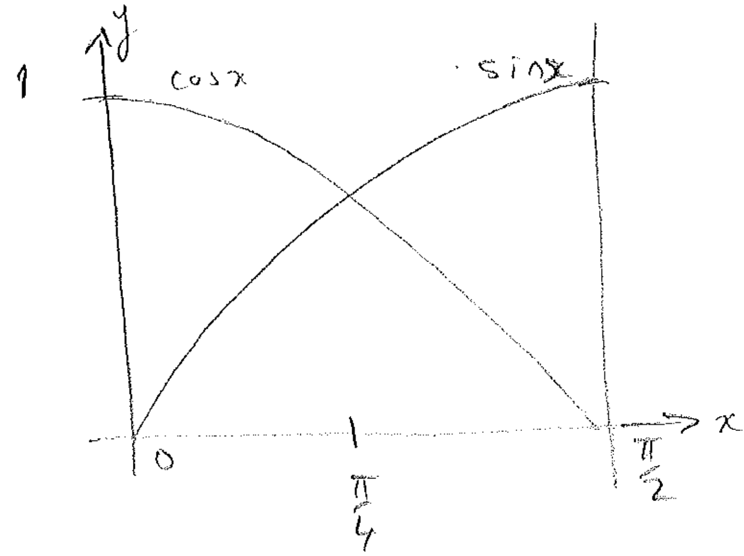
For $0 \le x \le 1$, $x^3 \le x^2$, so

$$A = \int_0^1 (x^2 - x^3)\,dx = \left[\frac{x^3}{3} - \frac{x^4}{4}\right]_0^1$$

$$= \frac{1}{3} - \frac{1}{4} - 0$$

$$= \frac{1}{12}.$$

4.



$\sin x = \cos x$

$\Rightarrow \quad \tan x = 1 \quad \Rightarrow \quad x = \frac{\pi}{4}$ for a solution in $[0, \frac{\pi}{2}]$.

For $0 \le x \le \frac{\pi}{4}$, $\sin x \le \cos x$

$\frac{\pi}{4} \le x \le \frac{\pi}{2}$, $\cos x \le \sin x$

Hence

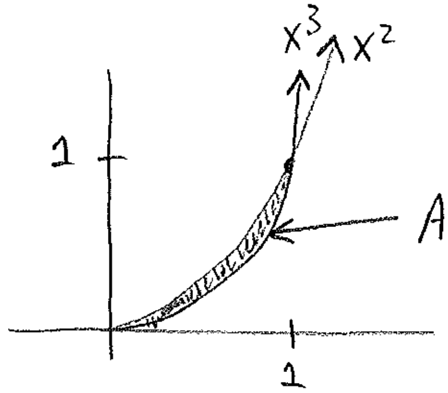
$$A = \int_0^{\frac{\pi}{4}} (\cos x - \sin x)\,dx + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} (\sin x - \cos x)\,dx$$

$$= \Big[\sin x + \cos x \Big]_0^{\frac{\pi}{4}} + \Big[-\cos x - \sin x \Big]_{\frac{\pi}{4}}^{\frac{\pi}{2}}$$

$$= \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} - (0+1) + \left(-0 - 1 - \left(-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\right)\right)$$

$$= \frac{4}{\sqrt{2}} - 1 - 1 = 2(\sqrt{2} - 1).$$

(#3)

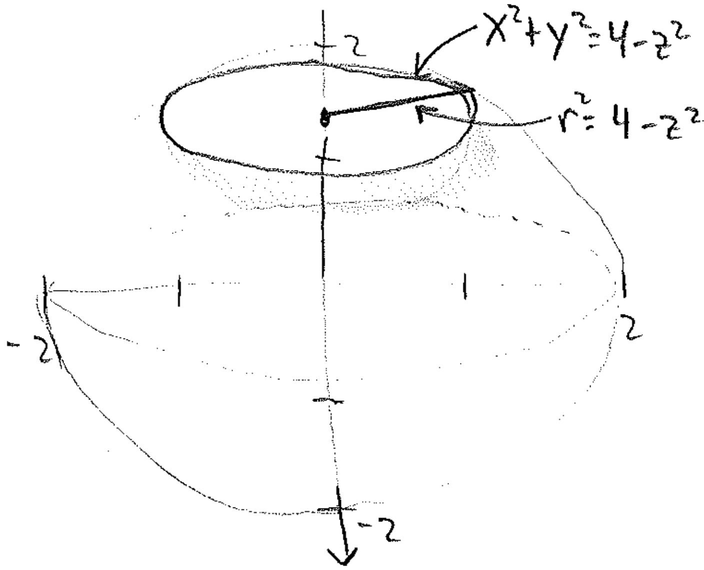


$$A = \int_0^1 (x^2 - x^3)\,dx$$

$$= \frac{1}{3}x^3 - \frac{1}{4}x^4 \Big|_0^1 = \frac{1}{3} - \frac{1}{4} = \frac{1}{12}$$

(#5)

$$x^2 + y^2 + z^2 = 2^2, \quad \text{so} \quad x^2 + y^2 = 4 - z^2$$



$$V = \int_1^2 \pi(4 - z^2)\,dz$$

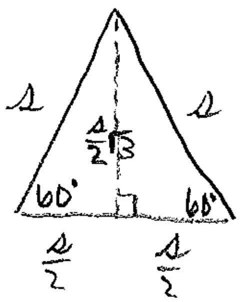
$$= \int_1^2 (4\pi - \pi z^2)\,dz$$

$$= 4\pi z - \frac{1}{3}\pi z^3 \Big|_1^2$$

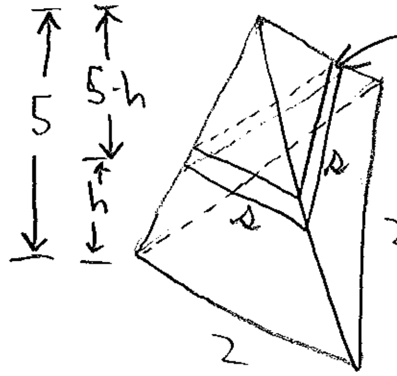
$$= 8\pi - \frac{8\pi}{3} - \left(4\pi - \frac{1}{3}\pi\right)$$

$$= 4\pi - \frac{7\pi}{3} = \frac{5\pi}{3}$$

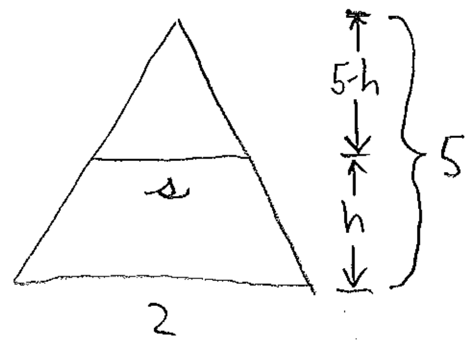
#6 Area of an equilateral Δ given side s.



$$A_\Delta = \frac{1}{2}\cdot b \cdot h = \frac{1}{2} s \cdot \frac{s}{2}\sqrt{3} = \frac{s^2\sqrt{3}}{4}$$



Each slice is an equilateral Δ



Find s as a function of h using similar triangles.

$$\frac{s}{2} = \frac{5-h}{5} \Rightarrow s = \frac{2(5-h)}{5}$$

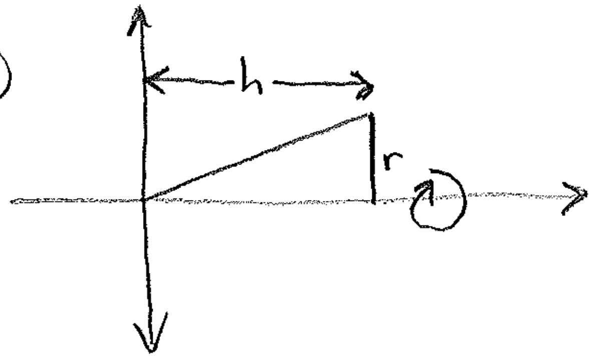
$$\text{Area}_\Delta = \frac{s^2\sqrt{3}}{4} = \frac{\left[\frac{2}{5}(5-h)\right]^2\sqrt{3}}{4} = \frac{\sqrt{3}}{25}(5-h)^2$$

$$\text{Vol} = \frac{\sqrt{3}}{25}\int_0^5 (25-10h+h^2)\,dh$$

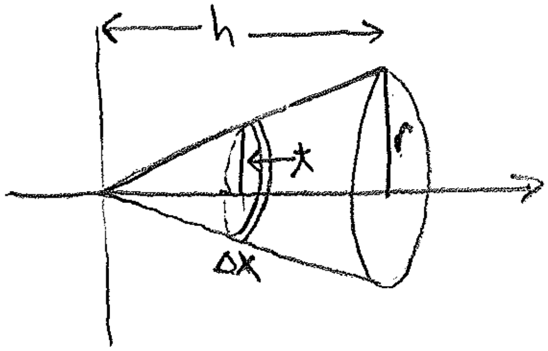
$$= \frac{\sqrt{3}}{25}\left(25h - 5h^2 + \frac{1}{3}h^3\Big|_0^5\right)$$

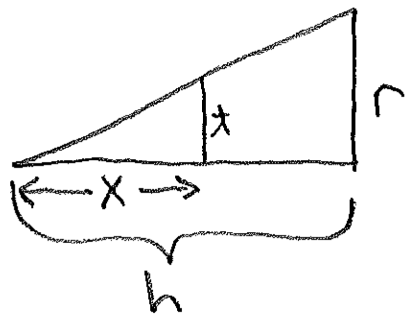
$$= \frac{\sqrt{3}}{25}\cdot\left(\cancel{125} - \cancel{125} + \frac{125}{3}\right) = \frac{5\sqrt{3}}{3} = \frac{1}{3}\cdot A_{base}\cdot \text{height}$$

#7



rotate the Δ around the x-axis





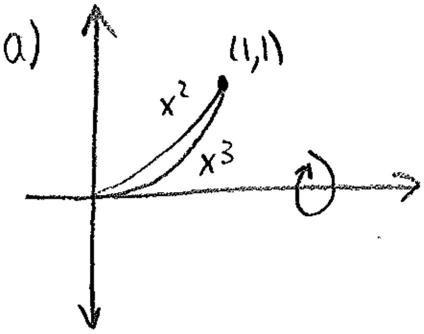
$$\frac{t}{r} = \frac{x}{h} \Rightarrow t = \frac{r}{h} \cdot x$$

r, h are constants

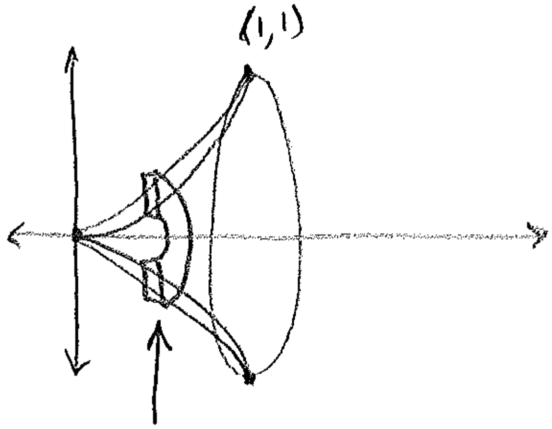
$$\text{Vol} = \int_0^h \pi \left(\frac{r}{h} x\right)^2 dx$$

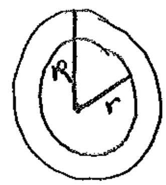
$$= \frac{\pi r^2}{h^2} \cdot \int_0^h x^2 \, dx = \frac{\pi r^2}{h^2} \cdot \left(\frac{1}{3} x^3 \Big|_0^h \right) = \frac{\pi r^2 h^3}{3 h^2} = \frac{1}{3} \pi r^2 h$$

#8 a)



Rotate around the x-axis.

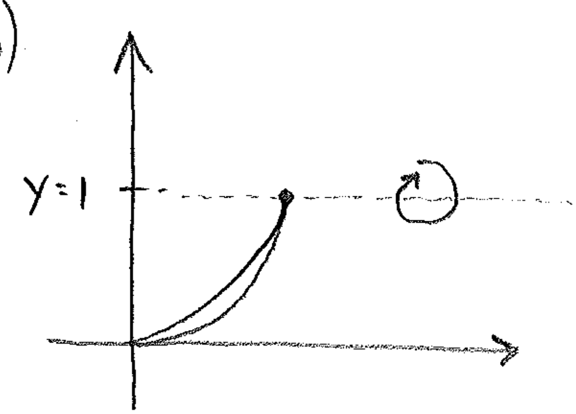


Each cross section looks like 

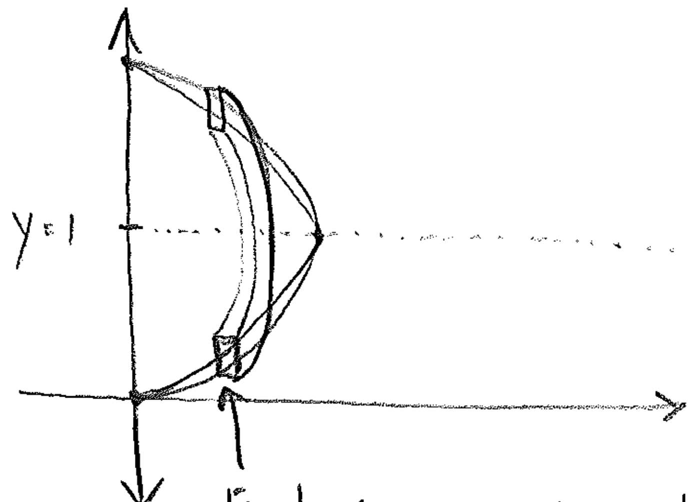
$$
\begin{aligned}
A &= \pi R^2 - \pi r^2 \\
&= \pi (R^2 - r^2) \\
&= \pi \left((x^2)^2 - (x^3)^2 \right) \\
&= \pi (x^4 - x^6)
\end{aligned}
$$

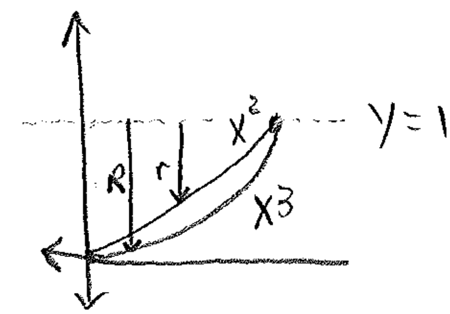
$$
\begin{aligned}
\text{Volume} &= \pi \int_0^1 (x^4 - x^6)\, dx \\
&= \pi \left(\tfrac{1}{5} x^5 - \tfrac{1}{7} x^7 \Big|_0^1 \right) \\
&= \pi \left(\tfrac{1}{5} - \tfrac{1}{7} \right) = \frac{2\pi}{35}
\end{aligned}
$$

#7 (b)

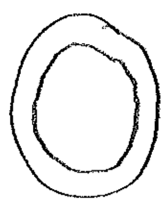


Rotate around line y=1





Each cross section looks like



$$A = \pi (R^2 - r^2)$$

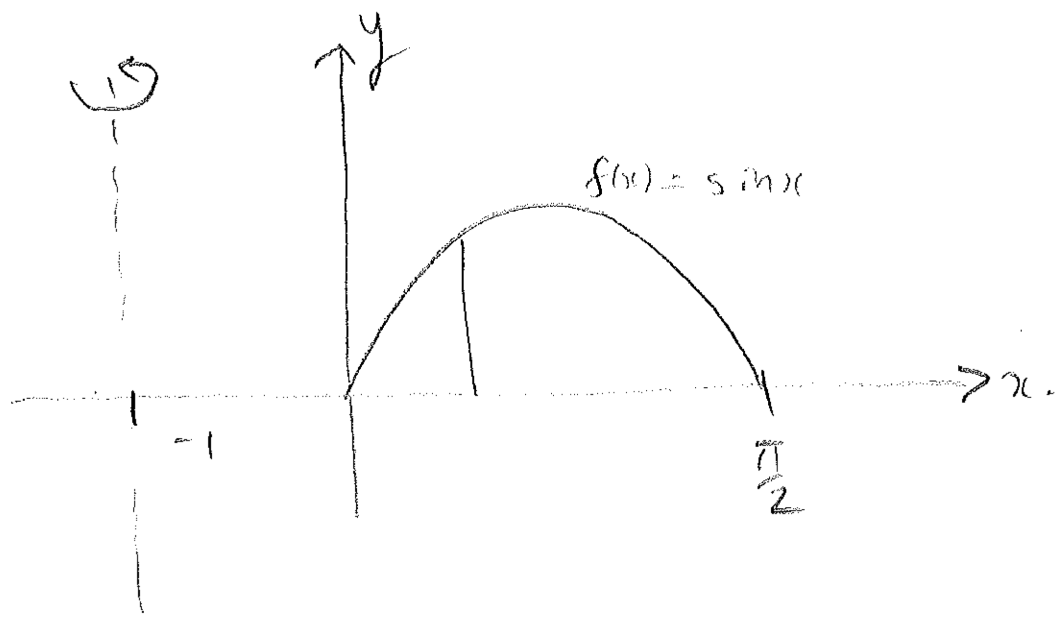
$$= \pi\left((1-x^3)^2 - (1-x^2)^2 \right)$$

$$= \pi\left(x^6 - x^4 - 2x^3 + 2x^2 \right)$$

$$\text{Vol} = \pi \int_0^1 (x^6 - x^4 - 2x^3 + 2x^2)\, dx$$

$$= \frac{23\pi}{210}$$

9.



Height of shell : $f(x) = \sin x$

Distance of shell to axis of rotation $x - (-1) = x + 1.$

$$V = \int_0^{\frac{\pi}{2}} 2\pi \text{ radius} \times \text{height } dx$$

$$= \int_0^{\frac{\pi}{2}} 2\pi (x+1) \sin x \, dx$$

$$= 2\pi \int_0^{\frac{\pi}{2}} (x+1) \sin x \, dx$$

Use integration by parts with

$u = x+1 \qquad dv = \sin x\,dx$

$du = dx \qquad v = -\cos x$

$$= 2\pi\left\{\left[-(x+1)\cos x\right]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} -\cos x\,dx\right\}$$

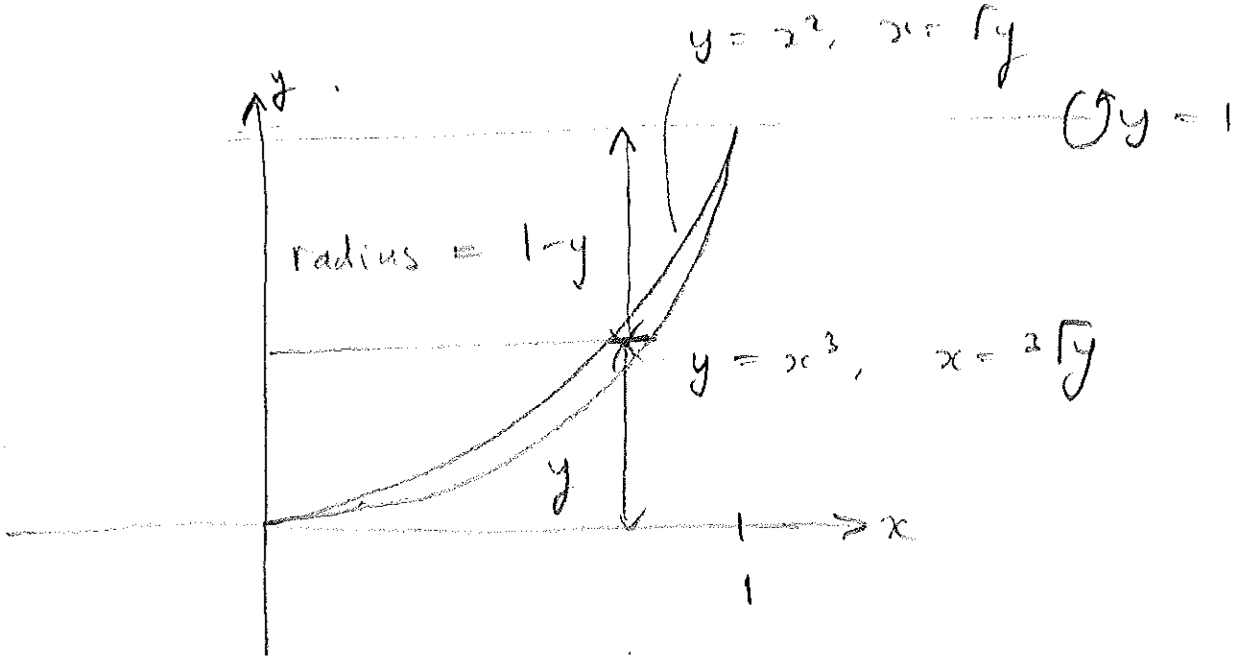
$$= 2\pi\left\{\left[-(x+1)\cos x\right]_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}} \cos x\,dx\right\}$$

$$= 2\pi\left\{\left[-(x+1)\cos x\right]_0^{\frac{\pi}{2}} + \left[\sin x\right]_0^{\frac{\pi}{2}}\right\}$$

$$= 2\pi\left(\left(0-(-1)\right)+(1-0)\right)$$

$$= 4\pi.$$

10.



Horizontal axis of rotation, so for shell method we need an integral in y.

Have 2 curves $x = \sqrt[3]{y}$, $x = \sqrt{y}$.

Intersect curves to find limits of integration

$$\sqrt[3]{y} = \sqrt{y}$$

$$y^{\frac{1}{3}} = y^{\frac{1}{2}}$$

$$(y^{\frac{1}{3}})^6 = (y^{\frac{1}{2}})^6$$

$$y^2 = y^3$$

$$y^2 - y^3 = 0$$

$$y^2(1-y) = 0$$

So $y = 0, 1$ and

$c = 0, \quad d = 1.$

For $0 \le y \le 1$, $\sqrt{y} \le \sqrt[3]{y}$, so the height of the shell is

$f(y) = \sqrt[3]{y} - \sqrt{y}$

The radius of the shell is the distance from y to the axis of rotation which is $1 - y$.

Hence

$$V = \int_c^d 2\pi \text{ radius} \times \text{height } dy$$

$$= \int_0^1 2\pi (1-y)(\sqrt[3]{y} - \sqrt{y})\, dy$$

$$= 2\pi \int_0^1 \left(y^{\frac{1}{3}} - y^{\frac{1}{2}} - y^{4/3} + y^{3/2} \right) dy$$

$$= 2\pi \left[\frac{3}{4} y^{4/3} - \frac{2}{3} y^{3/2} - \frac{3}{7} y^{7/3} + \frac{2}{5} y^{5/2} \right]_0^1$$

$$= 2\pi \left(\frac{3}{4} - \frac{2}{3} - \frac{3}{7} + \frac{2}{5} - 0 \right)$$

$$= 2\pi \left(\frac{315}{420} - \frac{280}{420} - \frac{180}{420} + \frac{168}{420} - 0 \right)$$

$$= 2\pi \left(\frac{483 - 460}{420} \right)$$

$$= 2\pi \left(\frac{23}{420} \right)$$

$$= \frac{23\pi}{210} \qquad (\text{as before } !)$$