Mth 141 Worksheet # 3 Solutions.

1. i) $\dfrac{1}{(x-3)(x+7)}$ PFE: $\dfrac{A}{x-3} + \dfrac{B}{x+7}$

ii) $\dfrac{2x+5}{(x-3)^2(x-7)^3}$

PFE: $\dfrac{A}{(x-3)} + \dfrac{B}{(x-3)^2} + \dfrac{C}{(x-7)} + \dfrac{D}{(x-7)^2} + \dfrac{E}{(x-7)^3}$

iii) $\dfrac{x^3-14x^2+7x+12}{(x-3)(x+7)}$ PFE: $Ax + B$

$+ \dfrac{C}{x-3} + \dfrac{D}{x+7}$

(long division needed).

iv) $\dfrac{x^2+7x+2}{(x^2+4)^2(x-1)^2}$

PFE: $\dfrac{Ax+B}{x^2+4} + \dfrac{Cx+D}{(x^2+4)^2} + \dfrac{E}{x-1} + \dfrac{F}{(x-1)^2}$

2. i) $\int \frac{dx}{x^2-1}$

$$\frac{1}{x^2-1} = \frac{1}{(x-1)(x+1)} = \frac{A}{x-1} + \frac{B}{x+1}$$

$$= \frac{A(x+1) + B(x-1)}{(x-1)(x+1)}.$$

So $$\frac{1}{(x-1)(x+1)} = \frac{(A+B)x + A - B}{(x-1)(x+1)}$$

Comparing like powers of x.

$x/$ $\quad A + B = 0 \quad$ ①

$1/$ $\quad A - B = 1 \quad$ ②

① $\Rightarrow$ $B = -A$

Sub in ② $\quad A - (-A) = 1$

$2A = 1$

$A = \frac{1}{2}$ so $B = -A = -\frac{1}{2}$.

$A = \frac{1}{2}$, $B = -\frac{1}{2}$.

Thus

$$\frac{1}{x^2-1} = \frac{1}{2(x-1)} - \frac{1}{2(x+1)}$$

and

$$\int \frac{dx}{x^2-1} = \int \left\{ \frac{1}{2(x-1)} - \frac{1}{2(x+1)} \right\} dx$$

$$= \frac{1}{2} \int \frac{dx}{x-1} - \frac{1}{2} \int \frac{dx}{x+1}$$

$$= \frac{1}{2} \ln|x-1| - \frac{1}{2} \ln|x+1| + C.$$

ii) $\displaystyle\int \frac{x^2+2x-1}{(x-1)(x+1)^2}\,dx.$

Write.

$$\frac{x^2+2x-1}{(x-1)(x+1)^2} = \frac{A}{x-1} + \frac{B}{x+1} + \frac{C}{(x+1)^2}.$$

$$= \frac{A(x+1)^2 + B(x-1)(x+1) + C(x-1)}{(x-1)(x+1)^2}$$

$$= \frac{A(x^2+2x+1) + B(x^2-1) + C(x-1)}{(x-1)(x+1)^2}.$$

$$\frac{x^2+2x-1}{(x-1)(x+1)^2} = \frac{(A+B)x^2 + (2A+C)x + A-B-C}{(x-1)(x+1)^2}.$$

Compare like powers of x.

x^2 $\quad A + B = 1 \quad$ (1)

x $\quad 2A + C = 2 \quad$ (2)

1 $\quad A - B - C = -1 \quad$ (3)

Add (1) & (3) to get rid of B

$2A - C = 0 \quad$ (4)

Add (2) and (4) to get rid of C

$$4A = 2$$

$$A = \frac{1}{2}.$$

Substitute for A in (1) to get B

$$\frac{1}{2} + B = 1$$

$B = \frac{1}{2}$

Substitute for A in (2) to get C

$$2\left(\tfrac{1}{2}\right) + C = 2$$

$$C = 1.$$

So $A = \frac{1}{2}$, $B = \frac{1}{2}$, $C = 1$ and the partial fractions expansion is

$$\frac{x^2 + 2x - 1}{(x-1)(x+1)^2} = \frac{1}{2(x-1)} + \frac{1}{2(x+1)} + \frac{1}{(x+1)^2}$$

and $$\int \frac{x^2+2x-1}{(x-1)(x+1)^2}\,dx = \int \left\{ \frac{1}{2(x-1)} + \frac{1}{2(x+1)} + \frac{1}{(x+1)^2} \right\} dx$$

$$= \frac{1}{2}\int \frac{dx}{x-1} + \frac{1}{2}\int \frac{dx}{x+1} + \int \frac{dx}{(x+1)^2}$$

$$= \frac{1}{2}\ln|x-1| + \frac{1}{2}\ln|x+1| - \frac{1}{x+1} + C.$$

iii) $\displaystyle\int \frac{2x^2}{(x-1)(x^2+1)} \cdot dx$

x^2+1 is irreducible and the P.F.E is

$$\frac{2x^2}{(x-1)(x^2+1)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+1}$$

$$= \frac{A(x^2+1) + Bx(x-1) + C(x-1)}{(x-1)(x^2+1)}$$

$$= \frac{A(x^2+1) + B(x^2-x) + C(x-1)}{(x-1)(x^2+1)}$$

$$\frac{2x^2}{(x-1)(x^2+1)} = \frac{(A+B)x^2 + (-B+C)x + A-C}{(x-1)(x^2+1)}.$$

Compare like powers of x.

x^2 $\quad A + B = 2 \quad (1)$

x $\quad -B + C = 0 \quad (2)$

1 $\quad A - C = 0 \quad (3)$

Add (1) and (2) to get rid of B

$$A + C = 2 \quad (4)$$

Add (3) and (4) to get rid of C

$$2A = 2$$

$$A = 1$$

Subst for A in (1) to get B

$$1 + B = 2$$

$$B = 1$$

Subst for A in ③ to get C

$$1 - C = 0$$
$$C = 1$$

So $A = B = C = 1$ and we have P.F.E.

$$\frac{2x^2}{(x-1)(x^2+1)} = \frac{1}{x-1} + \frac{x+1}{x^2+1}.$$

and

$$\int \frac{2x^2}{(x-1)(x^2+1)}\,dx = \int \left\{ \frac{1}{x-1} + \frac{x+1}{x^2+1} \right\} dx$$

$$= \int \frac{dx}{x-1} + \int \frac{x\,dx}{x^2+1} + \int \frac{dx}{x^2+1}$$

For the second integral let

$$u = x^2+1$$
$$du = 2x\,dx$$
$$\frac{du}{2} = x\,dx$$

$$= \int \frac{dx}{x-1} + \int \frac{\frac{du}{2}}{u} + \int \frac{dx}{x^2+1}$$

$$= \int \frac{dx}{x-1} + \frac{1}{2} \int \frac{du}{u} + \int \frac{dx}{x^2+1}$$

$$= \ln|x-1| + \frac{1}{2} \ln|u| + \arctan x + C$$

$$= \ln|x-1| + \frac{1}{2} \ln|x^2+1| + \arctan x + C .$$

3. i) $$\int_{-\infty}^{0} e^{4x}\,dx$$

Look at $$\int_a^0 e^{4x}\,dx = \left[e^{4a}/4\right]_a^0$$

$$= \frac{e^0}{4} - \frac{e^{4a}}{4}$$

$$\rightarrow \frac{e^0}{4} = \frac{1}{4} \quad \text{as} \quad a \rightarrow -\infty.$$

Hence $\int_{-\infty}^{0} e^{4x}\,dx$ converges and has value $\frac{1}{4}$.

ii) $$\int_2^{\infty} \frac{1}{x\sqrt{\ln x}}\,dx$$

Look at $\int_2^b \frac{1}{x\sqrt{\ln x}}\,dx$ and let $b \rightarrow \infty$.

<u>Substitution</u> : let $u = \ln x$, $du = \frac{dx}{x}$

<u>Limits</u> : When $x = 2$, $u = \ln 2$
$x = b$, $u = \ln b$

$$= \int_{\ln 2}^{\ln b} \frac{1}{\sqrt{u}}\, du$$

$$= \int_{\ln 2}^{\ln b} u^{-\frac{1}{2}}\, du$$

$$= \left[\frac{u^{\frac{1}{2}}}{\frac{1}{2}} \right]_{\ln 2}^{\ln b}$$

$$= \left[2\sqrt{u} \right]_{\ln 2}^{\ln b} = 2\sqrt{\ln b} - 2\sqrt{\ln 2}$$

As $b \to \infty$, $\ln b \to \infty$ and $\sqrt{\ln b} \to \infty$.

Hence $\displaystyle\int_2^{\infty} \frac{dx}{x\sqrt{\ln x}}$ diverges.

iii) $\displaystyle\int_0^{\frac{\pi}{2}} \tan x \, dx$

$$= \int_0^{\frac{\pi}{2}} \frac{\sin x}{\cos x} \, dx$$

Problem here is that $\cos(\frac{\pi}{2}) = 0$ and so we need to look at

$$\int_0^b \frac{\sin x}{\cos x} \, dx \quad \text{and let } b \to \frac{\pi}{2}- .$$

<u>Substitution</u> Let $u = \cos x$, $du = -\sin x \, dx$

<u>Limits</u>: When $x = 0$, $u = \cos 0 = 1$
$x = b$, $u = \cos b$

$$= \int_1^{\cos b} -\frac{du}{u}$$

$$= \int_{\cos b}^1 \frac{du}{u}$$

(get rid of minus sign by flipping the limits of integration.

$$= \left[\ln u\right]_{\cos b}^{1}$$

$$= \ln 1 - \ln(\cos b)$$

$$= -\ln(\cos b) \qquad (\ln 1 = 0).$$

As $b \to \frac{\pi}{2}-$, $\cos b \to 0_+$

and so $-\ln(\cos b) \to -(-\infty) = +\infty$.

Hence

$$\int_0^{\frac{\pi}{2}} \tan x \, dx$$

<u>diverges</u>.

iv) $$\int_{-\infty}^{\infty} x e^{x}\, dx$$

Need to split up into

$$\int_{-\infty}^{0} x e^{x}\, dx \quad \text{and} \quad \int_{0}^{\infty} x e^{x}\, dx$$

and investigate each piece separately.

For $$\int_{-\infty}^{0} x e^{x}\, dx$$

we look at

$$\int_{a}^{0} x e^{x}\, dx$$

and let $a \to -\infty$.

For $\displaystyle\int_a^0 x e^x \, dx$

use integration by parts with

$$u = x \,, \quad dv = e^x \, dx$$

$$du = dx \,, \quad v = e^x$$

$$= \left[x e^x \right]_a^0 - \int_0^a e^x \, dx$$

$$= \left[x e^x \right]_a^0 - \left[e^x \right]_a^0$$

$$= 0 - a e^a - (e^0 - e^a)$$

$$= -1 - e^a - a e^a$$

As $a \to -\infty$, $e^a \to 0$ and faster than any polynomial can grow so that $a e^a \to 0$ also (you may treat this as a known fact or derive it for yourself using l'Hôpital's rule).

Hence

$$\int_a^0 x e^x \, dx$$

$$\longrightarrow \quad -1 - 0 - 0 = -1$$

as $a \to -\infty$ and

$$\int_{-\infty}^0 x e^x \, dx \quad \text{converges to } -1.$$

For $\int_0^\infty x e^x \, dx$, we look at

$$\int_0^b x e^x \, dx \quad \text{and let } b \to \infty.$$

We can re-use the same antiderivative we found earlier using integration by parts:

$$\int_0^b x e^x \, dx = \left[x e^x - e^x \right]_0^b$$

$$= be^b - e^b - (0 - e^0)$$

$$= (b-1)e^b + 1$$

As $b \to \infty$, $e^b \to \infty$ as does $b-1$, so

$$\int_0^\infty x e^x \, dx$$

diverges.

Finally, since one of the 'pieces' does not converge, we have that

$$\int_{-\infty}^{\infty} x e^x \, dx$$

must also diverge.