Mth 142 Worksheet #2 Solutions

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1. i) $\int \sin^4 x \cos^5 x \, dx$

Power of cosine is odd

$$= \int \sin^4 x \cos^4 x \cos x \, dx$$

$$= \int \sin^4 x \, (1 - \sin^2 x)^2 \cos x \, dx$$

$u = \sin x$, $du = \cos x \, dx$

$$= \int u^4 (1-u^2)^2 \, du$$

$$= \int u^4 (1 - 2u^2 + u^4) \, du$$

$$= \int (u^4 - 2u^6 + u^8) \, du$$

$$= \frac{u^5}{5} - \frac{2u^7}{7} + \frac{u^9}{9} + C$$

$$= \frac{\sin^5 x}{5} - \frac{2\sin^7 x}{7} + \frac{\sin^9 x}{9} + C$$

ii) $\int \sin^2 x \cos^4 x \, dx$

Both powers even → use half-angle

$$= \int \frac{1}{2}(1-\cos 2x)\left(\frac{1}{2}(1+\cos 2x)\right)^2 dx$$

$$= \frac{1}{8}\int (1-\cos 2x)(1+2\cos 2x+\cos^2 2x)\, dx$$

$$= \frac{1}{8}\int (1+2\cos 2x+\cos^2 2x - \cos 2x$$
$$- 2\cos^2 2x - \cos^3 2x)\, dx$$

$$= \frac{1}{8} \int (1 + \cos 2x - \cos^2 2x - \cos^3 2x)\, dx$$

$$= \frac{1}{8} \left\{ \int (1 + \cos 2x)\, dx - \int \cos^2 2x \, dx \right.$$

$$\left. - \int \cos^3 2x \, dx \right\}$$

Use half-angle on $\cos^2 2x$
and the substitution $u = \sin 2x$
on the last integral $du = 2 \cos 2x$

$$= \frac{1}{8} \left\{ \int (1 + \cos 2x)\, dx - \int \frac{1}{2} (1 + \cos 4x)\, dx \right.$$

$$\left. - \int (1 - \sin^2 2x) \cos 2x \, dx \right\}$$

$$= \frac{1}{8} \left\{ \int (1 + \cos 2x)\, dx - \frac{1}{2} \int (1 + \cos 4x)\, dx \right.$$

$$\left. - \int (1 - u^2) \frac{du}{2} \right\}$$

$$= \frac{1}{8}\int (1+\cos 2x)\,dx - \frac{1}{16}\int (1+\cos 4x)\,dx$$

$$+ \frac{1}{16}\int (u^2-1)\,du$$

$$= \frac{1}{8}\left(x + \frac{\sin 2x}{2}\right) - \frac{1}{16}\left(x + \frac{\sin 4x}{4}\right)$$

$$+ \frac{1}{16}\left(\frac{u^3}{3} - u\right) + C$$

$$= \frac{x}{16} + \cancel{\frac{\sin 2x}{16}} - \frac{\sin 4x}{64}$$

$$+ \frac{\sin^3 2x}{48} - \cancel{\frac{\sin 2x}{16}} + C$$

$$= \frac{x}{16} - \frac{\sin 4x}{64} + \frac{\sin^3 2x}{48} + C.$$

iv) $\int \tan^3 x \sec x \, dx$

Power of tangent is <u>odd</u>

$$= \int (\sec^2 x - 1) \sec x \tan x \, dx$$

$u = \sec x$, $du = \sec x \tan x \, dx$

$\tan^2 x = \sec^2 x - 1$

$$= \int (u^2 - 1) \, du$$

$$= \frac{u^3}{3} - u + C$$

$$= \frac{\sec^3 x}{3} - \sec x + C$$

iii) $\int \tan^4 x \sec^2 x \, dx$

Power of secant is <u>even</u>

$u = \tan x, \quad du = \sec^2 x \, dx$

$= \int u^4 \, du$

$= \frac{u^5}{5} + C$

$= \frac{\tan^5 x}{5} + C.$

2 i). $\int \sqrt{4-x^2}\, dx.$

Let $x = 2\sin\theta$, $dx = 2\cos\theta\, d\theta$, $\theta = \arcsin\left(\frac{x}{2}\right)$

$$\begin{aligned}
\sqrt{4-x^2} &= \sqrt{4-4\sin^2\theta} \\
&= \sqrt{4(1-\sin^2\theta)} \\
&= 2\sqrt{1-\sin^2\theta} \\
&= 2\sqrt{\cos^2\theta} \\
&= 2\cos\theta .
\end{aligned}$$

Rewrite

$$\begin{aligned}
\int \sqrt{4-x^2}\, dx &= \int 2\cos\theta \cdot 2\cos\theta\, d\theta \\
&= 4\int \cos^2\theta\, d\theta .
\end{aligned}$$

$$= 4\int \frac{1}{2}(1+\cos 2\theta)\,d\theta$$

Using the half-angle identity

$$\cos^2\theta = \frac{1}{2}(1+\cos 2\theta)$$

$$= 2\int (1+\cos 2\theta)\,d\theta$$

$$= 2\left(\theta + \frac{\sin 2\theta}{2}\right) + C$$

$$= 2\theta + \sin 2\theta + C$$

$$= 2\arcsin\frac{x}{2} + \sin\left(2\arcsin\left(\frac{x}{2}\right)\right) + C$$

Can also say

$$\sin 2\theta = 2\sin\theta\cos\theta$$

$$= 2\sin\theta\sqrt{1-\sin^2\theta}$$

$$= x\sqrt{1-(x/2)^2} = x\sqrt{1-x^2/4}$$

So we can rewrite the answer as

$$2\arcsin\frac{x}{2} + x\sqrt{1-x^2/4} + C.$$

ii) $\int \sqrt{9+4x^2}\,dx = \int \sqrt{3^2+(2x)^2}\,dx$

Use tangent subst:

$2x = 3\tan\theta$, $\theta = \arctan\left(\frac{2x}{3}\right)$.

$2dx = 3\sec^2\theta\,d\theta$

$dx = \frac{3}{2}\sec^2\theta\,d\theta$

$\sqrt{9+4x^2} = \sqrt{9+9\tan^2\theta}$

$= 3\sqrt{1+\tan^2\theta}$

$= 3\sqrt{\sec^2\theta}$ using $\sec^2\theta = 1+\tan^2\theta$.

$= 3\sec\theta$.

Rewrite.

$\int \sqrt{9+4x^2}\,dx = \int 3\sec\theta \cdot \frac{3}{2}\sec^2\theta\,d\theta$

$= \frac{9}{2}\int \sec^3\theta\,d\theta.$

No need to go any further than this!
(Although it can be done by parts + partial fractions).

iii)
$$\int \frac{1}{\sqrt{4x-x^2}}\,dx.$$

Complete the square.

$$4x - x^2 = -(x^2 - 4x)$$
$$= -(x^2 + 2(-2)x)$$
$$= -(x^2 + 2(-2)x + (-2)^2 - (-2)^2)$$
$$= -((x-2)^2 - 4)$$
$$= 4 - (x-2)^2$$

So
$$\int \frac{1}{\sqrt{4x-x^2}}\,dx = \int \frac{1}{\sqrt{4-(x-2)^2}}\,dx.$$

Now let $x - 2 = 2\sin\theta$.

Then $\theta = \arcsin\left(\frac{x}{2} - 1\right)$

$dx = 2\cos\theta\, d\theta$

and $\sqrt{4-(x-2)^2} = \sqrt{4-4\sin^2\theta} = 2\cos\theta$.

Rewrite the integral as.

$$\int \frac{1}{2\cos\theta} \cdot 2\cos\theta \, d\theta$$

$$= \int d\theta$$

$$= \theta + C$$

$$= \arcsin\left(\frac{x}{2} - 1\right) + C.$$

iv) $$\int \frac{1}{(10+4x+4x^2)^{3/2}}\,dx$$

Complete the square.

$$4x^2+4x+10 = 4\left(x^2+x+\frac{10}{4}\right)$$

$$= 4\left(x^2+2\left(\tfrac{1}{2}\right)x+\left(\tfrac{1}{2}\right)^2-\left(\tfrac{1}{2}\right)^2+\frac{10}{4}\right)$$

$$= 4\left(\left(x+\tfrac{1}{2}\right)^2+\frac{9}{4}\right)$$

$$= (2x+1)^2+3^2.$$

Rewrite

$$\int \frac{1}{(10+4x+4x^2)^{3/2}}\,dx = \int \frac{1}{((2x+1)^2+3^2)^{3/2}}\,dx$$

Suggests tan subst.

$$2x+1 = 3\tan\theta$$

$$\theta = \arctan\left(\frac{2x+1}{3}\right).$$

$$2\,dx = 3\sec^2\theta\,d\theta$$

$$dx = \frac{3}{2}\sec^2\theta\,d\theta$$

$$((2x+1)^2+3^2)^{\frac{3}{2}}$$

$$= (9\tan^2\theta + 9)^{\frac{3}{2}}$$

$$= 9^{\frac{3}{2}}(1+\tan^2\theta)^{\frac{3}{2}}$$

$$= 9^{\frac{3}{2}}(\sec^2\theta)^{\frac{3}{2}} \qquad \text{as} \quad \sec^2\theta = 1+\tan^2\theta$$

$$= 27\sec^3\theta.$$

Rewrite.

$$\int \frac{1}{((2x+1)^2+3^2)^{\frac{3}{2}}}\,dx$$

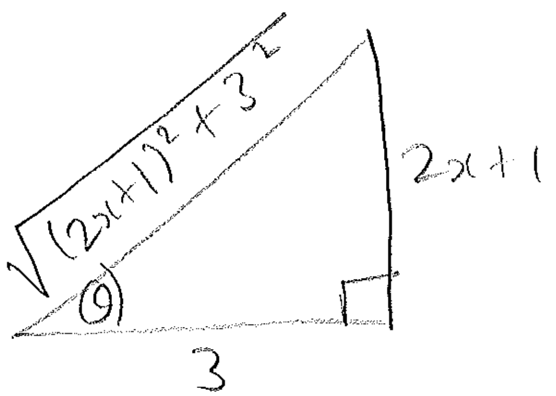
$$= \int \frac{1}{27\sec^3\theta}\cdot\frac{3}{2}\sec^2\theta\,d\theta$$

$$= \frac{1}{18} \int \frac{d\theta}{\sec\theta}$$

$$= \frac{1}{18} \int \cos\theta \, d\theta$$

$$= \frac{1}{18} \sin\theta + C$$

Use a right-angled triangle to figure out $\sin\theta$ in terms of x using $\tan\theta = \frac{2x+1}{3}$.



By Pythageras and defn of sine

$$\sin\theta = \frac{opp}{hyp}$$

$$= \frac{2x+1}{\sqrt{(2x+1)^2+3^2}}$$

$$= \frac{2x+1}{\sqrt{10+4x+4x^2}}$$

Then finally (!)

$$\int \frac{dx}{(10 + 4x + 4x^2)^{3/2}}$$

$$= \frac{1}{18} \sin \theta + C$$

$$= \frac{1}{18} \cdot \frac{2x+1}{\sqrt{10 + 4x + 4x^2}} + C$$

$$= \frac{2x+1}{18\sqrt{10 + 4x + 4x^2}} + C$$