§9.9 Convergence of Taylor Series.

Suppose we have some fn f which has derivatives of all orders at a given point x_0.

For x near x_0, recall that we had the Taylor remainder

$$R_n(x) = f(x) - P_n(x)$$

$$= f(x) - \sum_{k=0}^{n} f^{(k)}(x_0)(x - x_0)^k$$

This leads to the fact that

$$f(x) = \sum_{k=0}^{\infty} f^{(k)}(x_0)(x - x_0)^k$$

if and only if $\lim_{n \to \infty} R_n(x) = 0$.

In other words

f equals its Taylor series at x iff the remainder $R_n(x) \longrightarrow 0$ as $n \longrightarrow \infty$.

In practice it is hard to calculate the remainder $R_n(x)$ directly and we usually use the Lagrange error bound instead.

Recall that if $|f^{(n+1)}(t)| \leq M$ for t between x_0 and x, then we have

$$|R_n(x)| \leq \frac{M}{(n+1)!} |x - x_0|^{n+1}$$

Hence if we know the upper bound from Lagrange $\longrightarrow 0$ as $n \longrightarrow \infty$, then so must $R_n(x)$ by the squeeze principle.

Ex Find binomial series for

a) $\frac{1}{(1+x)^2}$, b) $\frac{1}{\sqrt{1+x}}$

a) $\frac{1}{(1+x)^2} = (1+x)^{-2}$

Use the binomial series to start writing out terms until a pattern (hopefully) emerges

$$= 1 + (-2)x + \frac{(-2)(-3)}{2!}x^2$$

$$+ \frac{(-2)(-3)(-4)}{3!}x^3 + \frac{(-2)(-3)(-4)(-5)}{4!}x^4$$

$$= 1 - 2x + \frac{3!}{2!}x^2 - \frac{4!}{3!}x^3 + \frac{5!}{4!}x^4 - \cdots$$

$$= 1 - 2x + 3x^2 - 4x^3 + 5x^4 -$$

$$= \sum_{k=0}^{\infty} (-1)^k (k+1) x^k$$ valid for $-1 < x < 1$.

b) $\frac{1}{\sqrt{1+x}} = (1+x)^{-\frac{1}{2}}$

$$= 1 - \frac{x}{2} + \frac{(-\frac{1}{2})(-\frac{3}{2})}{2!} x^2$$

$$+ \frac{(-\frac{1}{2})(-\frac{3}{2})(-\frac{5}{2})}{3!} x^3$$

$$+ \frac{(-\frac{1}{2})(-\frac{3}{2})(-\frac{5}{2})(-\frac{7}{2})}{4!} x^4$$

$$+ \cdots$$

$$= 1 - \frac{1}{2} x + \frac{(1)(3)}{2^2 \, 2!} x^2$$

$$- \frac{(1)(3)(5)}{2^3 \, 3!} x^3 + \frac{(1)(3)(5)(7)}{2^4 \, 4!} x^4$$

Pattern?

Powers of 2 and factorial in the bottom both match the power of x

ie 2^k, $k! \longrightarrow x^k$

On top, for x^k we have a product of <u>odd</u> numbers up to and including $2k-1$

e.g. $2(3)-1 = 5$ for x^3 term

$2(4)-1 = 7$ for x^4 term.

However, the pattern doesn't work for the constant term ($2(0)-1$ is negative) and so we have to list this term separately.

Finally, we also have a sign flip which is

$+1$ for even powers of x

-1 for odd powers of x.

Putting it all together . . . . . .

$$\frac{1}{\sqrt{1+x}} = 1 + \sum_{k=1}^{\infty} (-1)^k \frac{(1)(3)(5)\cdots(2k-1)}{2^k \, k!} x^k$$

valid for $-1 < x < 1$.