§ 9.8 Taylor Series

Taylor series are the power series we obtain from some function $f(x)$ near some pt. $x=a$ when we let the degree n of the Taylor polynomials P_n tend to infinity.

For example, we saw in the last section for $f(x) = e^x$ near $x=0$, we had.

$$e^x \approx P_n(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!}$$

It turns out that the larger we make n, the better $P_n(x)$ is as an approximation to e^x. In fact, in the limit as $n \to \infty$, $P_n(x)$ converges to e^x and we have

$$e^x = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots = \sum_{k=0}^{\infty} \frac{x^n}{k!}$$

The same thing happens with the Taylor polynomials for $\sin x$ and $\cos x$ and we get

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots + (-1)^{k-1}\frac{x^{2k+1}}{(2k+1)!} + \cdots$$

$$= \sum_{k=0}^{\infty} (-1)^{k+1} \frac{x^{2k+1}}{(2k+1)!}$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots + (-1)^k \frac{x^{2k}}{(2k)!} + \cdots$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!}$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^k}{k!} + \cdots$$

$$= \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

<u>Note:</u> These Taylor series converge to their respective f^ns for every value of x.

Taylor Series in General

Suppose $f(x)$ is infinitely differentiable at $x = x_0$ (i.e. $f^{(k)}(x_0)$ exists for every $k \geq 0$).

In this case, we can form the Taylor series in a similar way to the Taylor polynomials. However, the Taylor series may not converge to $f(x)$ for every value of x.

If the series does converge to $f(x)$, then

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \cdots + \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k + \cdots$$

$$= \sum_{k=0}^{\infty} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k.$$

When $x_0 = 0$, we get

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(k)}(0)}{k!}x^k + \cdots$$

$$= \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!}x^k$$

which is sometimes called the McLaurin series for f.

As we noted before, the Taylor series for f is a power series and the partial sums for this power series are the Taylor polynomials.

The two related questions we need to ask are

1. Where does the Taylor series converge?

2. Where does the Taylor series converge to $f(x)$?

Power Series in $x - x_0$

If $c_0, c_1, c_2, \ldots$ are constants, $x_0 \in \mathbb{R}$ and x is a variable, then an infinite series of the form

$$\sum_{k=0}^{\infty} c_k (x - x_0)^k$$

$$= c_0 + c_1 (x - x_0) + c_2 (x - x_0)^2 + \cdots + c_k (x - x_0)^k + \cdots$$

is called a power series in $(x - x_0)$ or a power series about x_0.

Examples $\displaystyle\sum_{k=0}^{\infty} x^k = 1 + x + x^2 + \cdots$

$$\sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2!} + \cdots$$

Any Taylor or McLaurin series

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k = f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \cdots$$

Radius and Interval of Convergence.

Each different value of x in a power series gives us a different infinite series.

The question then is for which x do we get a convergent series and for which x do we get a divergent series.

Note that if we set $x = x_0$, then $(x-x_0)^k = 0$ $\forall k \geq 1$ and so the series is just

$$c_0$$

which is trivially convergent

Using the radio or limit comparison tests, we can prove the following.

<u>Theorem</u> For any power series in $x - x_0$, exactly one of the following is true:

a) The series converges only for $x = x_0$.

b) The series converges (absolutely) for all real values of x.

c) $\exists R > 0$ such that the series converges (absolutely) for $x \in (x_0 - R, x_0 + R)$ and diverges for $x < x_0 - R$ or $x > x_0 + R$.

At the endpoints $x_0 - R$, $x_0 + R$ the series may be absolutely convergent, conditionally convergent or divergent – depends on the particular series.

R is called the radius of convergence of the series and the set of values of x where the series converges is then an interval, called the interval of convergence.

In case a) we say the radius of convergence is 0 and the interval of convergence is $\{x_0\}$.

In case b) we say the radius of convergence is ∞ and the interval of convergence is all of $\mathbb{R}$ i.e. $(-\infty, \infty)$.

<u>Ex</u> a) $\sum_{k=0}^{\infty} x^k$

Let $u_k = x^k$

Apply the ratio test with

$$\frac{|u_{k+1}|}{|u_k|} = \frac{|x^{k+1}|}{|x^k|} = \left|\frac{x^{k+1}}{x^k}\right|$$

$$= |x|$$ (don't need to worry about $x = 0$ - why?)

$$\to |x|$$

as $k \to \infty$.

Series converges if $|x| < 1$, ie $-1 < x < 1$

and diverges if $|x| > 1$ ie $x < -1$

or $x > 1$.

Ratio test is inconclusive when $|x| = 1$.

When $x = 1$, we have $\sum_{k=0}^{\infty} 1 = 1 + 1 + 1 + \cdots$

$x = -1$ $\sum_{k=0}^{\infty} (-1)^k = 1 - 1 + 1 - 1 + \cdots$

and both of these diverge by the divergence test.

Hence the interval of convergence is $(-1, 1)$.

b) $\sum_{k=0}^{\infty} \frac{x^k}{k!}$

Let $u_k = \frac{x^k}{k!}$ and apply the ratio test

$$\frac{|u_{k+1}|}{|u_k|} = \frac{\left|\frac{x^{k+1}}{(k+1)!}\right|}{\left|\frac{x^k}{k!}\right|} = \left|\frac{x^{k+1}}{(k+1)!}\right| \cdot \left|\frac{k!}{x^k}\right|$$

$$= \frac{|x|^{k+1}}{|x^k|} \cdot \frac{k!}{(k+1)!}$$

$$= |x| \cdot \frac{\cancel{k}(\cancel{k-1}) \cdots \cancel{2} \cdot \cancel{1}}{(k+1)\cancel{k}(\cancel{k-1}) \cdots \cancel{2}\cdot 1}$$

$$= \frac{|x|}{k+1} \longrightarrow 0 \text{ as } k \to \infty.$$

By the ratio test, this series converges absolutely for every $x \in \mathbb{R}$ and the radius of convergence is ∞ while the interval of convergence is all of $\mathbb{R}$.

c) $\sum_{k=0}^{\infty} k!\, x^k$

Let $u_k = k!\, x^k$ and apply the ratio test

$$\frac{|u_{k+1}|}{|u_k|} = \frac{|(k+1)!\, x^{k+1}|}{|k!\, x^k|}$$

$$= \frac{(k+1)!}{k!} \frac{|x|^{k+1}}{|x|^k}$$

$$= (k+1)|x| \longrightarrow \infty \quad \text{as } k \to \infty$$
if $x \neq 0$.

Series diverges everywhere except at 0 and the radius of convergence is 0.

d) $$\sum_{k=0}^{\infty} \frac{(-1)^k x^k}{3^k (k+1)}$$

Let $u_k = \dfrac{(-1)^k x^k}{3^k (k+1)}$ and apply the ratio test using $|(-1)^{k+1}| = |(-1)^k| = 1$.

$$\left|\frac{u_{k+1}}{u_k}\right| = \frac{\left|\dfrac{(-1)^{k+1} x^{k+1}}{3^{k+1}((k+1)+1)}\right|}{\left|\dfrac{(-1)^k x^k}{3^k (k+1)}\right|}$$

$$= \frac{\dfrac{x^{k+1}}{3^{k+1}(k+2)}}{\dfrac{x^k}{3^k (k+1)}}$$

$$= \frac{x^{k+1}}{3^{k+1}(k+2)} \cdot \frac{3^k (k+1)}{x^k}$$

$$= \frac{x^{k+1}}{x^k} \cdot \frac{3^k}{3^{k+1}} \cdot \frac{k+1}{k+2}$$

$$= x \cdot \frac{1}{3} \cdot \frac{1 + \frac{1}{k}}{1 + \frac{2}{k}}$$

$$\longrightarrow \frac{x}{3} \quad \text{as} \quad k \to \infty.$$

Hence we have convergence if

$$\left|\frac{x}{3}\right| < 1$$

ie $|x| < 3$, $-3 < x < 3$

and divergence if $\left|\frac{x}{3}\right| > 1$

ie $|x| > 3$, $x < -3$ or $x > 3$.

Test at endpts. $x = -3, 3$.

$$x = -3: \quad \sum_{k=0}^{\infty} \frac{(-1)^k(-3)^k}{3^k(k+1)} = \sum_{k=0}^{\infty} \frac{3^k}{3^k(k+1)}$$

$$= \sum_{k=0}^{\infty} \frac{1}{k+1}$$

divergent (harmonic series).

$x = 3$: $$\sum_{k=0}^{\infty} \frac{(-1)^k 3^k}{3^k (k+1)} = \sum_{k=0}^{\infty} \frac{(-1)^k}{k+1}$$

conditionally convergent (alternating harmonic series).

Hence the radius of convergence is 3 and the interval of convergence is $(-3, 3]$.

Ex. $\ln x$ about $x=1$.

We showed in the last section that the Taylor polynomials are given by

$$\ln x \approx P_n(x) = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \cdots + (-1)^{n-1}\frac{(x-1)^n}{n}$$

A similar calculation gives the Taylor series

$$\ln x = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \cdots + (-1)^{n-1}\frac{(x-1)^k}{k} + \cdots$$

$$= \sum_{k=1}^{\infty} (-1)^{k-1}\frac{(x-1)^k}{k}.$$

If we then let $a_k = (-1)^{k-1}\frac{(x-1)^k}{k}$, then

$$\lim_{k\to\infty}\frac{|a_{k+1}|}{|a_k|} = \lim_{k\to\infty}\left|\frac{(-1)^k(x-1)^{k+1}}{k+1} \Big/ \frac{(-1)^{k-1}(x-1)^k}{k}\right|$$

$$= \lim_{k\to\infty}\left|(x-1)\cdot\frac{k}{k+1}\right|.$$

$$= |x-1| \lim_{k\to\infty} \left|\frac{k}{k+1}\right|$$

$$= |x-1| \lim_{k\to\infty} \left|\frac{1}{1+\frac{1}{k}}\right|$$

$$= |x-1|$$

By the ratio test, the Taylor series will then converge provided

$$|x-1| < 1$$

ie $\quad 0 < x < 2.$

When $x=0$, we have the series

$$-1 - \frac{1}{2} - \frac{1}{3} - \frac{1}{4} - \cdots = \sum_{k=1}^{\infty} -\frac{1}{k}$$

which diverges as it is a negative harmonic series

When $x=2$, we have the series.

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k}$$

which converges by the alternating series test.

Thus the series converges for $0 < x \leq 2$ (i.e. on the interval $(0,2]$) and diverges everywhere else.

In fact, on $(0,2]$, the Taylor series not only converges, but it also converges to $\ln x$.

The Binomial Series Expansion.

We find the Taylor series for $f(x) = (1+x)^p$ about $x=0$. Here p is a constant, but not necessarily a positive integer. Here

$f(x) = (1+x)^p$ so $f(0) = 1$, $C_0 = 1$

$f'(x) = p(1+x)^{p-1}$ $f'(0) = p$, $C_1 = p$

$f''(x) = p(p-1)(1+x)^{p-2}$ $f''(0) = p(p-1)$, $C_2 = \dfrac{p(p-1)}{2!}$

$f'''(x) = p(p-1)(p-2)(1+x)^{p-3}$ $f'''(0) = p(p-1)(p-2)$, $C_3 = \dfrac{p(p-1)(p-2)}{3!}$

etc.

Thus the Taylor series for f is

$$1 + px + \frac{p(p-1)x^2}{2!} + \frac{p(p-1)(p-2)x^3}{3!} + \cdots$$

$$\cdots + \frac{p(p-1)(p-2)\cdots(p-n+1)x^n}{n!} + \cdots$$

$$= 1 + \sum_{k=1}^{\infty} \frac{p(p-1)\cdots(p-k+1)}{k!} x^n$$

If we let $a_k = \frac{p(p-1)\cdots(p-k+1)}{k!} x^k$, then

$$a_{k+1} = \frac{p(p-1)\cdots(p-k+1)(p-(k+1)+1)}{(k+1)!} x^{k+1}$$

$$= \frac{p(p-1)\cdots(p-k+1)(p-k)}{(k+1)!}$$

So

$$\frac{|a_{k+1}|}{|a_k|} = \left| \frac{\dfrac{p(p-1)\cdots(p-k+1)(p-k)\,x^{k+1}}{(k+1)!}}{\dfrac{p(p-1)\cdots(p-k+1)\,x^k}{k!}} \right|$$

$$= \left| \frac{p(p-1)\cdots(p-k+1)(p-k)\cdot k!}{p(p-1)\cdots(p-k+1)\cdot(k+1)!} \, \frac{x^{k+1}}{x^k} \right|$$

$$= \left| \frac{\cancel{p(p-1)}\cdots\cancel{(p-k+1)}(p-k)\cdot \cancel{k(k-1)\cdots 2\cdot 1}\cdot x^{k+1}}{\cancel{p(p-1)}\cdots\cancel{(p-k+1)}(k+1)\cancel{k(k-1)\cdots 2\cdot 1}\cdot x^k} \right|$$

$$= \left| \frac{p-k}{k+1}\, x \right|$$

$$= \left| \frac{k-p}{k+1}\, x \right|$$

$$= \left| \frac{1-p/k}{1+1/k} \right| |x| \longrightarrow |x| \quad \text{as } k \to \infty.$$

By the ratio test, we then see that our binomial series converges (absolutely) if $|x|<1$ and diverges if $|x|>1$.

Again, if $|x|<1$ the series not only converges, but it converges to $(1+x)^p$ and so we can say that

$$(1+x)^p = 1 + px + \frac{p(p-1)}{2!}x^2 + \frac{p(p-1)(p-2)}{3!}x^3 + \cdots$$

$$+ \frac{p(p-1)\cdots(p-k+1)}{k!}x^n + \cdots, \quad |x|<1.$$

$$= 1 + \sum_{k=1}^{\infty} \frac{p\cdots(p-k+1)}{k!}x^k, \quad |x|<1.$$

What happens at the endpts. $x=\pm 1$ needs to be checked on an individual basis.

Note that if p is a natural number, then for $k > p$

$$p - k + 1 \leq 0$$

and so the Taylor coefficient

$$\frac{p(p-1)(p-2)\cdots(p-k+1)}{k!}$$

must contain 0 as one of the factors in the numerator and is $\therefore$ 0.

Thus, in this case the series terminates after the term in x^p and we get the familiar expansion for $(1+x)^p$ from the binomial theorem.

Ex. Use the binomial series with $p=4$ to expand $(1+x)^4$.

$$(1+x^4) = 1 + 4x + \frac{4\cdot 3}{2!}x^2 + \frac{4\cdot 3\cdot 2}{3!}x^3 + \frac{4\cdot 3\cdot 2\cdot 1}{4!}x^4.$$

All other terms have 0 as a factor and are $\therefore$ 0.

Thus

$$(1+x)^4 = 1 + 4x + 6x^2 + 4x^3 + x^4$$

which is the same as we'd get from the binomial theorem.

Ex. Find the Taylor series about $x = 0$ for $\frac{1}{1+x}$.

$$\frac{1}{1+x} = (1+x)^{-1} = 1 + (-1)x + \frac{(-1)(-2)}{2!}x^2 + \frac{(-1)(-2)(-3)}{3!}x^3$$

$$+ \frac{(-1)(-2)(-3)(-4)}{4!}x^4 + \cdots$$

$$= 1 - x + x^2 - x^3 + x^4 - \cdots$$

$$= \sum_{k=0}^{\infty} (-1)^k x^k \quad , \qquad -1 < x < 1 .$$