§ 9.7 McLaurin and Taylor Polynomials

The idea in this section is to find a way of choosing a polynomial which gives a good approximation of the values of some function $f(x)$ near a chosen point $x = x_0$.

We have already seen something of this in Math 141.

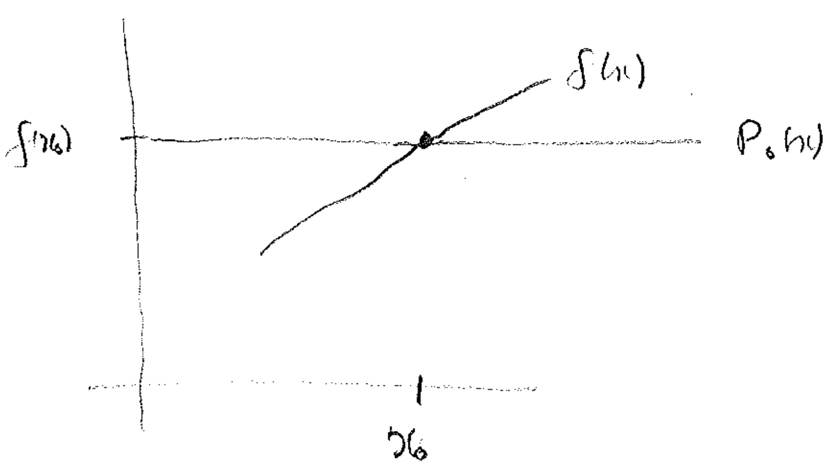
For example, if $f(x)$ is continuous (cts.) at $x = x_0$, then for x near x_0,

$$f(x) \approx f(x_0) \qquad \text{(ie } f(x) \text{ is close to } f(x_0)\text{)}$$

and so if we let $P_0(x)$ be the constant fn

$$P_0(x) = f(x_0)$$

then $P_0(x)$ gives us an approx. of f near $x = x_0$.



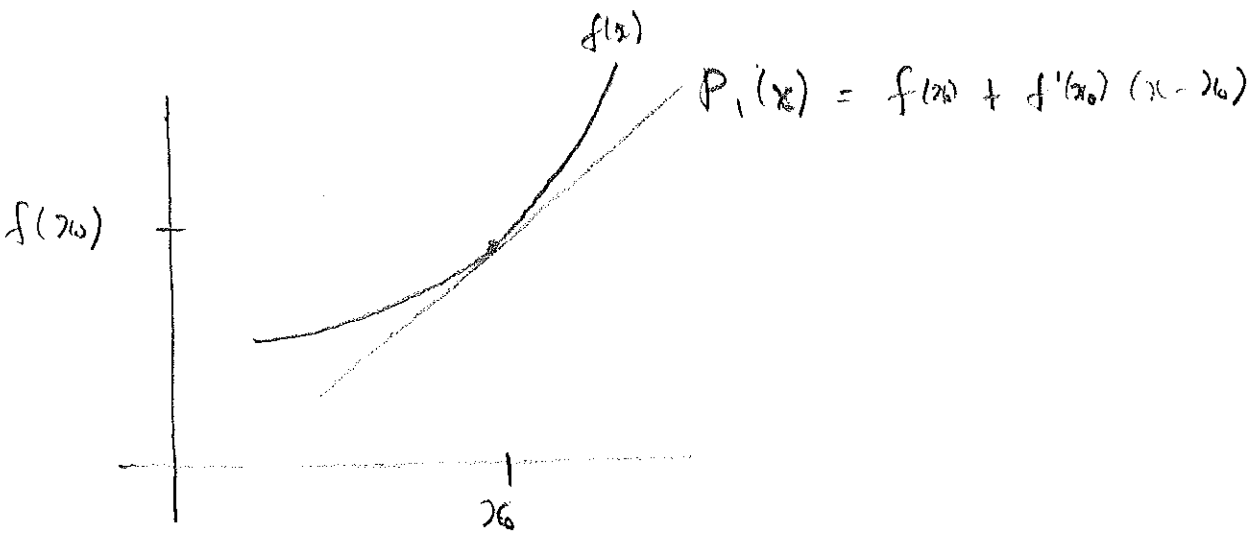
Note that $P_0(x)$ has the same value as $f(x)$ for $x = x_0$.

A better way to do the approximation is using linear f^{n}s.

This gives us the tangent line approximation also called the linear approximation.

Recall that if $f(x)$ is differentiable at $x = x_0$, then for x near x_0

$$f(x) \approx f(x_0) + f'(x_0)(x - x_0).$$



If we then let $P_1(x) = f(x_0) + f'(x_0)(x - x_0)$ then P_1 gives us an approx for f near $x = x_0$.

Note that $P_1(x)$ has the same <u>value</u> as $f(x)$ at $x = x_0$ and also the same <u>slope</u> as f at $x = x_0$.

<u>Ex.</u> Find the linear approx. for

$f(x) = \sin x$ near $x = 0$.

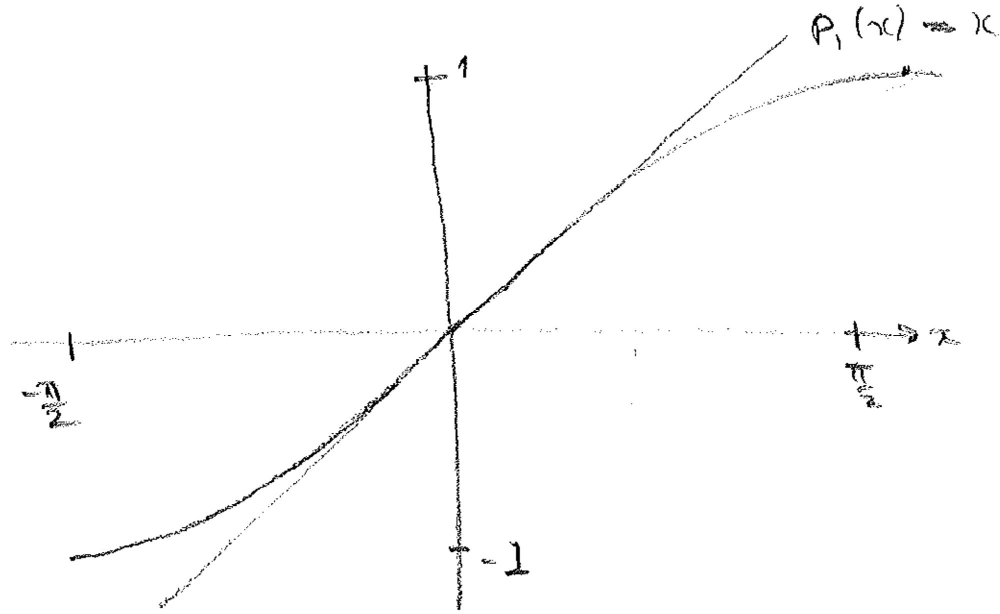
Here $f(x) = \sin x$, $f(0) = \sin(0) = 0$

$f'(x) = \cos x$, $f'(0) = \cos(0) = 1$

and so $P_1(x) = 0 + 1(x-0) = x$

and

$\sin x \approx x$, x small



e.g. $P_1(.2) = .2$ which is close to

$\sin(.2) = .1986\ldots\ldots$

Quadratic Approximations

Since a linear approx. is clearly better (more accurate) than a constant approx., it is natural to guess that we could do even better if we approximated $f(x)$ using quadratic polynomials

We choose our quadratic polynomial by extending the reasoning for the constant and linear approximations, namely we require that our quadratic have the same value, the same first derivative and the same second derivative as f does at $x = x_0$.

So suppose that f is twice differentiable at $x = x_0$, and let $P_2(x)$ be our quadratic approximation.

A useful trick here is instead of writing something like

$$P_2(x) = c_0 + c_1 x + c_2 x^2$$

(which is probably the first thing one might try), we instead take into account that we want an approximation near $x = x_0$ and write

$$P_2(x) = c_0 + c_1 (x - x_0) + c_2 (x - x_0)^2 .$$

Note that we already did something like this for the linear approximation.

Then

$$P_2(x) = C_0 + C_1(x-x_0) + C_2(x-x_0)^2,$$

$$P_2(x_0) = C_0$$

and so $P_2(x_0) = f(x_0) \Rightarrow \underline{C_0 = f(x_0)}$

$$P_2'(x) = C_1 + 2C_2(x-x_0),$$

$$P_2'(x_0) = C_1$$

and so $P_2'(x_0) = f'(x_0) \Rightarrow C_1 = f'(x_0)$.

$$P_2''(x) = 2C_2$$

$$P_2''(x_0) = 2C_2$$

and so $P_2''(x_0) = f''(x_0) \Rightarrow 2C_2 = f''(x_0)$, $C_2 = \dfrac{f''(x_0)}{2}$.

Get

$$P_2(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2}(x - x_0)^2$$

and so for x near x_0,

$$f(x) \approx f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2}(x - x_0)^2$$

Note how the constant and linear terms are the same as for the constant and linear approximations. In this sense we can say the quadratic approximation 'contains' the constant and linear approximations.

Ex. Find the quadratic approx. for

$f(x) = \cos x$ near $x = 0$.

Here $f(x) = \cos x$, $f(0) = \cos(0) = 1$

and so $c_0 = f(0) = 1$.

$f'(x) = -\sin x$, $f'(0) = -\sin(0) = 0$

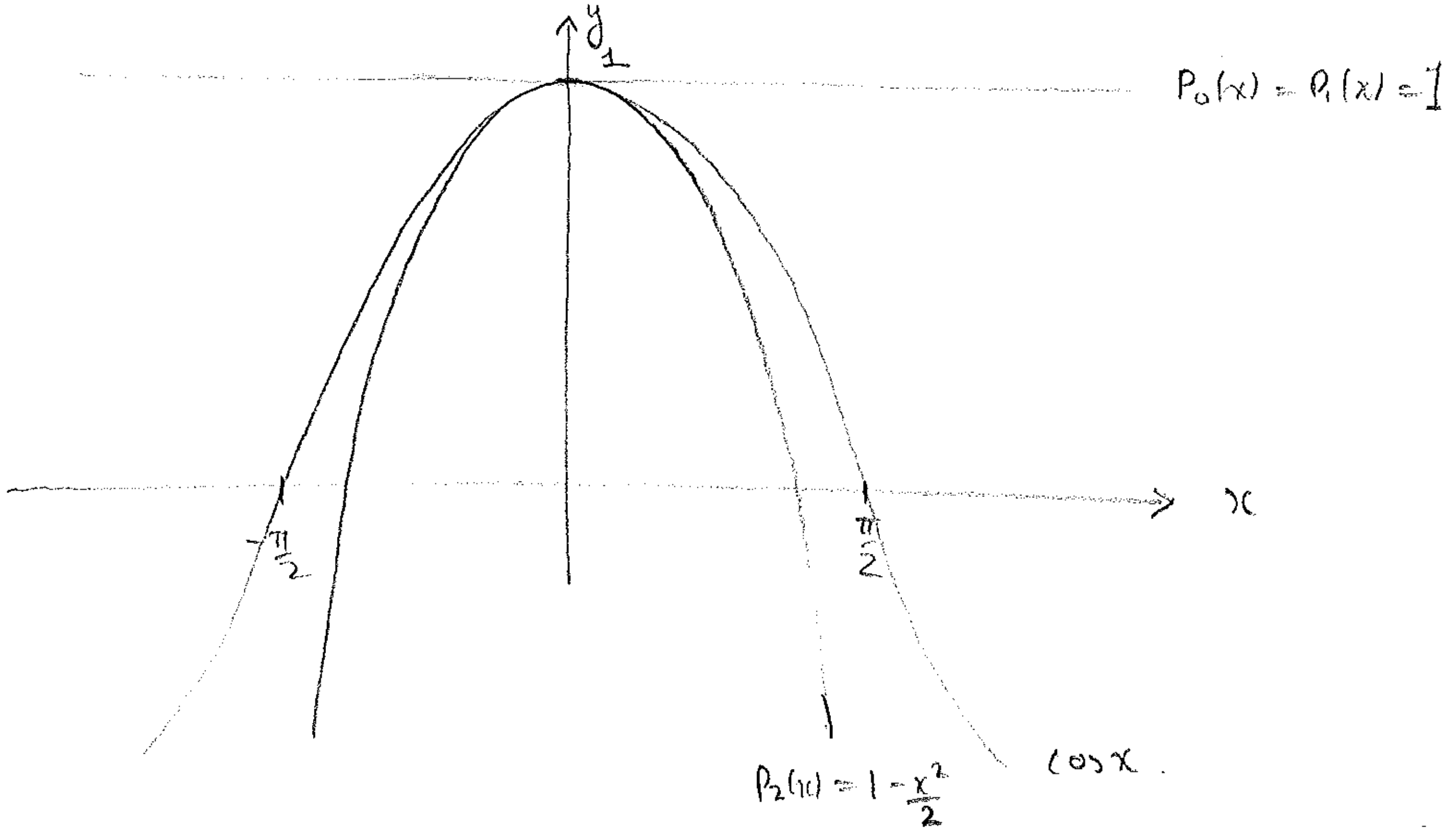
and so $c_1 = f'(0) = 0$

$f''(x) = -\cos x$, $f''(0) = -\cos(0) = -1$

and so $c_2 = \dfrac{f''(0)}{2} = -\dfrac{1}{2}$.

Thus $P_2(x) = 1 - \dfrac{x^2}{2}$ and

$\cos x \approx 1 - \dfrac{x^2}{2}$, x small.



e.g. $P_2(.4) = .92$

which is much closer to the actual value of $\cos(.4) = .921\ldots$ than

$P_0(.4) = P_1(.4) = 1$.

Approximation Using Higher Degree Polynomials

Suppose $f(x)$ is differentiable n times at $x = x_0$ and we try to approximate $f(x)$ near x_0 with a polynomial of degree n. Similarly to before we write this polynomial as

$$P_n(x) = C_0 + C_1(x-x_0) + C_2(x-x_0)^2 + C_3(x-x_0)^3 + \cdots + C_n(x-x_0)^n.$$

As before we want $P_n(x_0) = f(x_0)$, and so setting $x = x_0$, we get

$$P_n(x_0) = C_0 + 0 + 0 + \qquad + 0 = f(x_0)$$

So $C_0 = f(x_0)$. (same as before).

Now differentiate w.r.t. x to get

$$P_n'(x) = 0 + C_1 + 2C_2(x-x_0) + 3C_3(x-x_0)^2 + \qquad + nC_n(x-x_0)^{n-1}$$

Again, we want $P_n'(x_0) = f'(x_0)$ and so

$$P_n'(x_0) = C_1 + 0 + \quad + 0 = f'(x_0).$$

Thus $C_1 = f'(x_0)$ (again, same as before).

Differentiating again, we get:

$$P_n''(x) = 2C_2 + 3\cdot 2\, C_3(x - x_0) + \cdots + n(n-1)C_n(x-x_0)^{n-2}$$

We want $P_n''(x_0) = f''(x_0)$ and so

$$P_n''(x_0) = 2C_2 + 0 + \quad + 0 = f''(x_0).$$

Thus $2C_2 = f''(x_0)$

ie $C_2 = \dfrac{f''(x_0)}{2\cdot 1} = \dfrac{f''(x_0)}{2!}$ (again).

Now differentiate again to get

$$P_n'''(x) = 3 \cdot 2\, C_3 + \cdots + n(n-1)(n-2)\, C_n (x-x_0)^{n-3}$$

Similarly to above, we want $P_n'''(x_0) = f'''(x_0)$ and so

$$P_n'''(x_0) = 3 \cdot 2\, C_3 + 0 + \quad + 0 = f'''(x_0).$$

Thus $\quad 3 \cdot 2\, C_3 = f'''(x_0)$

or $\quad C_3 = \dfrac{f'''(x_0)}{3 \cdot 2 \cdot 1} = \dfrac{f'''(x_0)}{3!}$

Continuing in this way, we would find that

$$C_4 = \frac{f^{(4)}(x_0)}{4 \cdot 3 \cdot 2 \cdot 1} = \frac{f^{(4)}(x_0)}{4!}$$

and so on up to

$$C_n = \frac{f^{(n)}(x_0)}{n(n-1)(n-2)\cdots 2 \cdot 1} = \frac{f^{(n)}(x_0)}{n!}$$

To summarize.

If f is n times differentiable at $x = x_0$, then for x near x_0

$$f(x) \approx P_n(x)$$
$$= f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n$$

$P_n(x)$ is called the <u>Taylor polynomial of degree n</u> about $x = x_0$.

Since $0! = 1$, $1! = 1$, we can rewrite $P_n(x)$ as

$$P_n(x) = \frac{f(x_0)}{0!} + \frac{f'(x_0)}{1!}(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n$$

$$= \sum_{i=0}^{n} \frac{f^{(i)}(x_0)}{i!}(x-x_0)^i.$$

Ex. Find $P_7(x)$ for $f(x) = \sin x$ about $x = 0$.

Compare $P_7(\frac{\pi}{3})$ with $\sin(\frac{\pi}{3})$.

Have

$$f(x) = \sin x \;, \quad \text{gives} \quad f(0) = 0, \quad C_0 = \frac{0}{0!} = 0$$

$$f'(x) = \cos x \;, \quad f'(0) = 1, \quad C_1 = \frac{1}{1!} = 1$$

$$f''(x) = -\sin x \;, \quad f''(0) = 0, \quad C_2 = \frac{0}{2!} = 0$$

$$f'''(x) = -\cos x \;, \quad f'''(0) = -1 \;, \quad C_3 = \frac{-1}{3!}$$

$$f^{(4)}(x) = \sin x, \quad f^{(4)}(0) = 0, \quad C_4 = \frac{0}{4!} = 0$$

$$f^{(5)}(x) = \cos x, \quad f^{(5)}(0) = 1, \quad C_5 = \frac{1}{5!}$$

$$f^{(6)}(x) = -\sin x \quad f^{(6)}(0) = 0, \quad C_6 = \frac{0}{6!} = 0$$

$$f^{(7)}(x) = -\cos x \quad f^{(7)}(0) = -1, \quad C_7 = \frac{-1}{7!}$$

Thus, for x near 0

$$\sin x \approx P_7(x) = 0 + 1 \cdot x + 0x^2 - \frac{1}{3!}x^3 + 0x^4$$
$$+ \frac{1}{5!}x^5 + 0x^6 - \frac{1}{7!}x^7$$

$$= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}$$

At $\frac{\pi}{3} = 1.0471976\ldots$

$$P_7\left(\frac{\pi}{3}\right) = 0.8660213\ldots$$

while

$$\sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2} = 0.8660254\ldots$$

__Ex.__ Find $P_n(x)$ for $f(x) = e^x$ about $x = 0$ (any n).

Since $\frac{d}{dx}(f(x)) = \frac{d}{dx}(e^x) = e^x = f(x)$, for __any__ i

$$f^{(i)}(0) = e^0 = 1.$$

Thus $$C_i = \frac{f^{(i)}(0)}{i!} = \frac{1}{i!}$$

and so

$$P_n(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!}$$

and for x small

$$e^x \approx 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!}$$

Ex Find $P_n(x)$ for $f(x) = \frac{1}{1-x}$, x near 0.

$f(x) = \frac{1}{1-x}$, $f(0) = 1$, so $C_0 = 1$

$f'(x) = \frac{-1}{(1-x)^2}(-1)$

$= \frac{1}{(1-x)^2}$, $f'(0) = 1$, so $C_1 = 1$.

$f''(x) = \frac{-2}{(1-x)^3} \cdot (-1)$

$= \frac{2}{(1-x)^3}$, $f''(0) = 2$, so $C_2 = \frac{2}{2} = 1$.

$f'''(x) = \frac{-6}{(1-x)^4}(-1)$

$= \frac{6}{(1-x)^4}$, $f'''(0) = 6$, so $C_3 = \frac{6}{3!} = 1$.

Continuing in this way, we get $f^{(4)}(0) = 24 = 4!$, $f^{(5)}(0) = 120 = 5!$ and so on up to

$f^{(n)}(0) = n!$

Thus

$$C_i = \frac{f^{(i)}(0)}{i!} = \frac{i!}{i!} = 1$$

and so

$$P_n(x) = 1 + x + x^2 + x^3 + \qquad + x^n$$

and for x small

$$\frac{1}{1-x} \approx 1 + x + x^2 + \qquad + x^n.$$

Note that this is pretty much what we'd expect given that for $-1 < x < 1$, we have

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots + x^n + \cdots$$

<u>Ex</u> Find $P_n(x)$ for $f(x) = \ln x$ for x near 1.

$f(x) = \ln x$, so $f(1) = \ln 1 = 0 \Rightarrow c_0 = 0$

$f'(x) = \frac{1}{x}$, $f'(1) = \frac{1}{1} = 1 \Rightarrow c_1 = 1$.

$f''(x) = -\frac{1}{x^2}$, $f''(1) = -1 \Rightarrow c_2 = -\frac{1}{2!} = -\frac{1}{2}$

$f'''(x) = \frac{2}{x^3}$, $f'''(1) = 2 \Rightarrow c_3 = \frac{2}{3!} = \frac{1}{3}$

$f^{(4)}(x) = -\frac{6}{x^4}$, $f^{(4)}(1) = -6 \Rightarrow c_4 = -\frac{6}{4!} = -\frac{1}{4}$

Continuing in this way, we see that

$$f^{(i)}(1) = (-1)^{i-1} \cdot (i-1)!, \quad \text{so}$$

$$c_i = \frac{f^{(i)}(1)}{i!} = \frac{(-1)^{i-1}(i-1)!}{i!} = \frac{(-1)^{i-1}(i-1)(i-2)\cdots 2\cdot 1}{i\,(i-1)(i-2)\cdots 2\cdot 1}$$

$$= \frac{(-1)^{i-1}}{i}.$$

Thus

$$P_n(x) = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \cdots + \frac{(-1)^{n-1}(x-1)^n}{n}.$$

and

$$\ln x \approx (x-1) - \frac{(x-1)^2}{2} + \qquad + \frac{(-1)^{n-1}(x-1)^n}{n},$$

x near 1.

The Error in Taylor Polynomial Approximations

Suppose we want to approximate the values of some fn $f(x)$ near $x = x_0$ using the n-th degree Taylor polynomial $P_n(x)$. In order to know whether we have a good approximation we need to look at the error or remainder

$$R_n(x) = f(x) - P_n(x).$$

Of course we want the size of $R_n(x)$ to be small and we do this by finding an upper bound for $|R_n|$. The idea then is that if this upper bound is small, then R_n has to also be small.

Suppose now that we know $|f^{(n+1)}(x)| \leq M$ for some constant M and for some range of non-negative values of x, say for $0 \leq x \leq \delta$. (case for negative values is similar)

Thus

$$-M \leq f^{(n+1)}(x) \leq M \qquad \text{for} \quad 0 \leq x \leq \delta.$$

Since $R_n^{(n+1)}(x) = f^{(n+1)}(x)$, we have

$$-M \leq R_n^{(n+1)}(x) \leq M \qquad \text{for} \quad 0 \leq x \leq \delta.$$

The idea is then to go from a bound on the $(n+1)$st derivative $R_n^{(n+1)}(x)$ to a bound on $R_n(x)$ by repeated integrations from 0 to x.

Thus

$$-\int_0^x M\,dt \leq \int_0^x R_n^{(n+1)}(t)\,dt \leq \int_0^x M\,dt, \qquad 0 \leq x \leq \delta$$

Finding an Error Bound

For the sake of simplicity, assume for now that $a = 0$ (the case $a \neq 0$ is very similar)

Recall that we constructed $P_n(x)$ so that the first n derivatives of P_n at $x = 0$ were the same as those of f.

Thus

$$R_n(0) = f(0) - P_n(0) = 0$$

$$R_n'(0) = f'(0) - P_n'(0) = 0$$

$$R_n''(0) = f''(0) - P_n''(0) = 0$$

$$\vdots$$

$$R_n^{(n)}(0) = f^{(n)}(0) - P^{(n)}(0) = 0$$

If we now take one more derivative, then, since $P_n(x)$ is a polynomial of degree n, $P_n^{(n+1)}(x) = 0$, and so

$$R_n^{(n+1)}(x) = f^{(n+1)}(x).$$

By the First Fundamental Theorem of Calculus, since $E_n^{(n)}(x)$ is an antid. of $E_n^{(n+1)}(x)$, this gives

$$\Big[-Mt\Big]_0^x \leq \Big[R_n^{(n)}(t)\Big]_0^x \leq \Big[Mt\Big]_0^x$$

$$-Mx - 0 \leq R_n^{(n)}(x) - R_n^{(n)}(0) \leq Mx - 0$$

and since $R_n^{(n)}(0) = f^{(n)}(0) - P_n^{(n)}(0) = 0$, we have

$$-Mx \leq R_n^{(n)}(x) \leq Mx \,, \qquad 0 \leq x \leq \delta$$

Now integrate again drom 0 to x

$$\int_0^x -Mt\,dt \leq \int_0^x R_n^{(n)}(t)\,dt \leq \int_0^x Mt\,dt$$

and by FToC I again

$$\left[-\frac{Mt^2}{2}\right]_0^x \leq \Big[R_n^{(n-1)}(t)\Big]_0^x \leq \left[\frac{Mt^2}{2}\right]_0^x$$

and since $R_n^{(n-1)}(0) = f^{(n-1)}(0) - P_n^{(n-1)}(0) = 0$,

$$-\frac{Mx^2}{2} \leq R_n^{(n-1)}(x) \leq \frac{Mx^2}{2}, \quad 0 \leq x \leq \delta$$

If we keep going like this, we eventually undo all the derivatives and in the end we get

$$-\frac{1}{(n+1)!} Mx^{n+1} \leq R_n(x) \leq \frac{1}{(n+1)!} Mx^{n+1}, \quad 0 \leq x \leq \delta.$$

Thus

$$|R_n(x)| = |f(x) - P_n(x)| \leq \frac{1}{(n+1)!} Mx^{n+1}, \quad 0 \leq x \leq \delta.$$

We then get.

The Lagrange Error Bound for $P_n(x)$

Suppose f and all its derivatives are cts.
If $P_n(x)$ is the nth Taylor polynomial
for $f(x)$ about $x = x_0$, then

$$|R_n(x)| = |f(x) - P_n(x)| \leq \frac{M}{(n+1)!} |x - x_0|^{n+1}$$

where $|f^{(n+1)}(t)| \leq M$ on the interval between
x_0 and x.

Using the Lagrange Error Bound

Ex. Give a bound on R_4 when e^x is approximated by $P_4(x)$ about $x = 0$ on the interval $-.5 \le x \le .5$.

Let $f(x) = e^x$.

Then $f^{(5)}(x) = e^x$ and since e^x is positive and increasing

$$|f^{(5)}(x)| \le e^{0.5} = \sqrt{e} < 2, \quad -.5 \le x \le .5.$$

Thus we can take $M = 2$ which gives

$$|R_4| = |e^x - P_4(x)| \le \frac{2}{5!}|x|^5, \quad -.5 \le x \le .5.$$

Thus, for $-.5 \le x \le .5$

$$e^x \approx 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!}$$

and the error is at most $\frac{2}{120}(.5)^5 < .00006$.

Of course, for smaller values of x, we get smaller errors.

e.g $\quad R_4(.2) \approx 2.75 \times 10^{-6}$

while $\quad R_4(0.1) \approx 8.47 \times 10^{-8}$

One can check that

$$R_4(.2)/32 \approx 8.62 \times 10^{-\infty} \approx E(.1).$$

This is very much what we'd expect given our bound

$$|R_4(x)| \leq \frac{1}{60} |x|^5$$

since decreasing the value of x by a factor of 2 will decrease the value of the bound by a factor of $2^5 = 32$.

Ex. Show that the Taylor series for $\cos x$ converges to $\cos x$ everywhere on $\mathbb{R}$.

Letting $f(x) = \cos x$

$$f^{(n)}(x) = \begin{cases} \cos x \\ -\sin x \\ -\cos x \\ \sin x \end{cases}, \quad \text{depending on the value of } n.$$

Thus $|f^{(n+1)}(x)| \leq 1$ on $\mathbb{R}$ and using the Lagrange error bound (on any interval about 0 containing x) with $M = 1$,

$$|R_n(x)| = |\cos x - P_n(x)| \leq \frac{|x|^{n+1}}{(n+1)!}$$

One can show that for any fixed $a > 0$

$$\lim_{n \to \infty} \frac{a^n}{n!} = 0$$

and hence $|R_n(x)| \to 0$ as $n \to \infty$.

Since x was an arbitrary real number, $P_n(x) \to \cos x$ as $n \to \infty$ everywhere on $\mathbb{R}$ as required.