§9.6 Alternating Series

These are series where the terms alternate in sign e.g.

$$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots + \frac{(-1)^{k+1}}{k} + \cdots$$

For these series we have the

Alternating Series Test

An alternating series of the form

$$\sum_{k=1}^{\infty} (-1)^{k-1} a_k = a_1 - a_2 + a_3 - a_4 + \cdots + (-1)^{k-1} a_k + \cdots$$

converges if

i) $0 \leq a_{k+1} \leq a_k$ for all k and

ii) $\lim_{k \to \infty} a_k = 0$.

Ex1 The previous series

$$\sum_{k=1}^{\infty} \frac{(-1)^k}{k}$$

is easily seen to satisfy the convergence criteria for the alternating series test and is thus convergent.

Note that this series is <u>not</u> absolutely convergent as

$$\sum_{k=1}^{\infty} \left| \frac{(-1)^k}{k} \right| = \sum_{k=1}^{\infty} \frac{1}{k} \qquad \text{(harmonic series)}$$

which we know is divergent.

A series such as $\sum_{k=1}^{\infty} \frac{(-1)^k}{k}$ which is convergent but not absolutely convergent is known as <u>conditionally convergent</u>.

Ex 2. $$\sum_{k=1}^{\infty} (-1)^{k+1} \frac{k+1}{k+2}$$

Here the terms in the series do alternate. However,

$$\lim_{k \to \infty} \frac{k+1}{k+2} = \lim_{k \to \infty} \frac{1 + \frac{1}{k}}{1 + \frac{2}{k}} = 1 \neq 0.$$

Hence this series is divergent by the divergence test.

This example shows that one needs to be careful with alternating series!

Behaviour of the Partial Sums of an Alternating Series.

Let $\sum_{k=1}^{\infty} (-1)^{k-1} a_k$ be an alternating series where $0 \leq a_{k+1} \leq a_k$ for every k and $\lim_{k \to \infty} a_k = 0$.

Let $S_n = \sum_{k=1}^{n} (-1)^{k-1} a_i$ be the n-th partial sum.

Then $S_1 = a_1 > 0$

$S_2 = a_1 - a_2 > 0$ as $a_2 < a_1$, but clearly also

$S_2 < a_1$ as $a_2 > 0$.

$S_3 = a_1 - a_2 + a_3 > a_1 - a_2 = S_2$ but

$a_1 - a_2 + a_3 = a_1 - (a_2 - a_3) < a_1 = S_1$

as $a_2 - a_3 > 0$ since $a_2 > a_3$.

So $\quad S_1 > 0, \quad 0 < S_2 < S_1,$

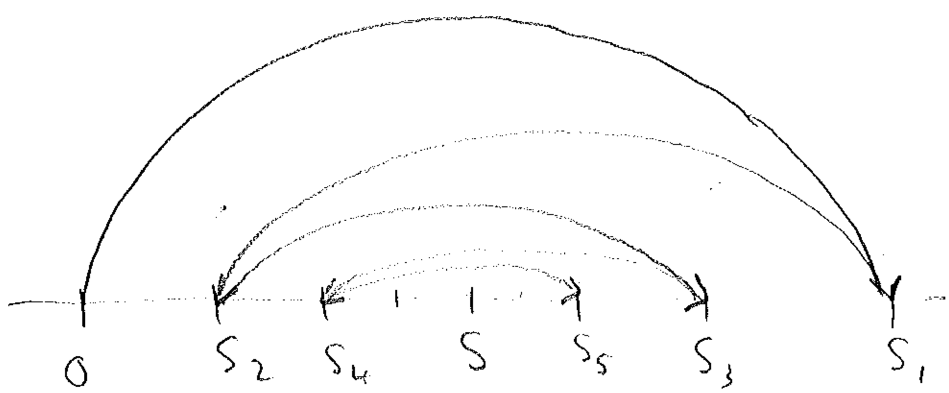
$S_2 < S_3 < S_1$ and this pattern

continues e.g. $\quad S_2 < S_4 < S_3$

$S_4 < S_5 < S_3$

etc.

Since $|S_n - S_{n-1}| = |a_n| \to 0$ as $n \to \infty$,
the partial sums oscillate inwards
to a fixed limit S



This is the basic idea behind the proof for our criterion for the convergence of alternating series.

We can also see that

$$S_2 < S_4 < S_6 < \quad < S < \quad S_5 < S_3 < S_1$$

Hence we have either

$$S_n < S < S_{n+1} \quad (n \text{ even})$$

<u>or</u> $$S_{n+1} < S < S_n \quad (n \text{ odd}).$$

Also, in either case

$$|S - S_n| < |S_{n+1} - S_n| = |a_n|.$$

This allows us to estimate the error involved in summing only finitely many terms of an alternating series.

Ex3 Estimate the error in approximating the sum of the alternating harmonic series

$$\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k}$$

by taking the sum of the first 9 terms.

$$S_9 = 1 - \frac{1}{2} + \frac{1}{3} - \cdots + \frac{1}{9} = 0.7456\ldots$$

The error involved in using S_9 as an approx. for S is then bounded by

$$|a_{10}| = \left|-\frac{1}{10}\right| = .1$$

Recognizing Series - Which Test to Use

Since there are so many tests available, I suggest the following rough guidelines for checking the convergence or divergence of series.

Steps for Testing a Series $\sum a_k$

1. Check $\lim_{k \to \infty} a_k$. If this isn't 0 or doesn't exist, then $\sum a_k$ is divergent by the divergence test.

2. Is the series alternating? If yes, then try the alternating series test.

3. If a_n contains factorials or powers involving k, e.g. $(2k)!$, 2^k, e^{-k^2}, etc, then try the ratio test or root test.

4. If a_k contains only fixed powers of k, e.g. k^2, $(k-1)^3$, $\sqrt{2k^2+6}$ etc., then try the comparison test or the limit comparison test.

   N.b. you need to have non-negative terms for the regular comparison test and positive terms for the limit comparison test.

DISCLAIMER

This system isn't foolproof and you need to be flexible.

As with methods of integration, if one test doesn't work, you need to try another.

Lastly, the book's summary of all the tests with guidelines (p. 645) is quite useful.