§ 9.5 The Comparison, Ratio and Root Tests

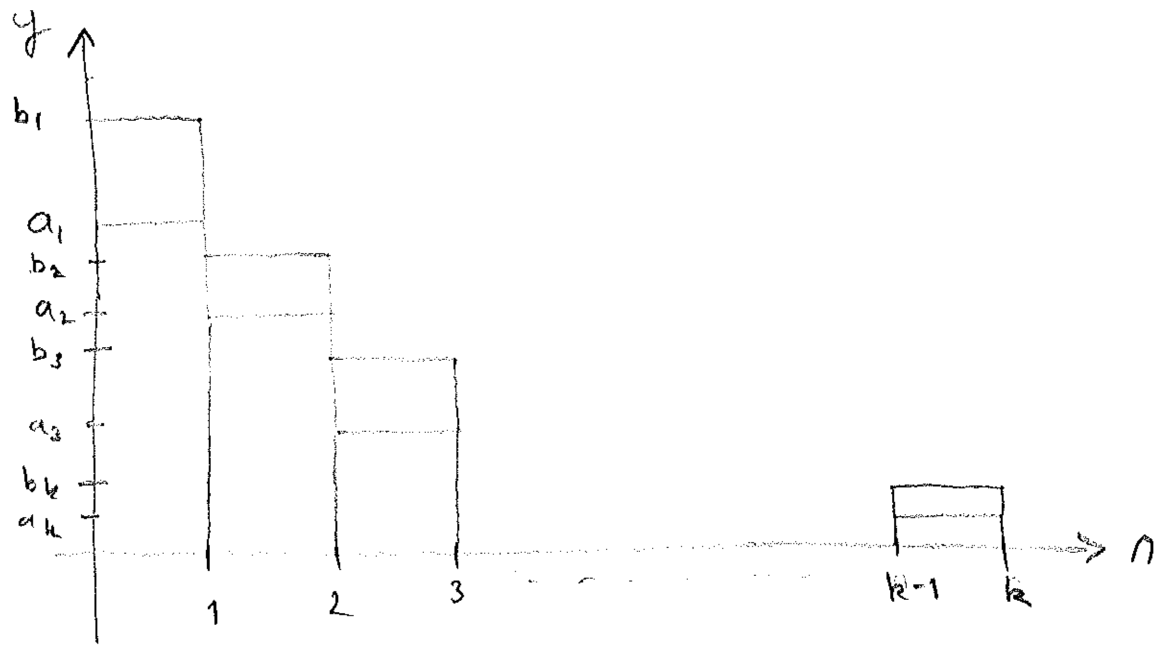
The Comparison Test

Suppose $0 \leq a_k \leq b_k$ for all k large enough.

i) If the bigger series $\sum b_k$ converges, then the smaller series $\sum a_k$ also converges

ii) If the smaller series $\sum a_k$ diverges, then the bigger series $\sum b_k$ also diverges.

The idea behind the proof can be best understood in terms of areas using the following picture



Over each interval $[k-1,k]$ we have a taller rectangle of height b_k and a shorter rectangle of height a_k.

This test is important not only in itself, but because it is used to derive other tests, in particular the ratio and nth root tests.

Ex $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^3+1}$

Since $k^3 \leq k^3+1$ for $k \geq 1$,

$$0 \leq \frac{1}{k^3+1} \leq \frac{1}{k^3} \quad \text{for } k \geq 1.$$

$\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^3}$ is a p-series with $p = 3 > 1$ and so convergent. Thus $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^3+1}$ is also convergent by the comparison test.

Ex $\displaystyle\sum_{k=1}^{\infty} \frac{k-1}{k^3+3}$

The convergence or divergence of this series is determined when n is large. For large n,

$$\frac{k-1}{k^3+3} \approx \frac{k}{k^3} = \frac{1}{k^2}$$

This suggests that we should compare with $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^2}$ and as this series converges, we should expect our series to converge also.

Since a fraction increases if its numerator is made larger or its denominator is made smaller, we have

$$0 \leq \frac{k-1}{k^3+3} \leq \frac{k}{k^3+3} \leq \frac{k}{k^3} = \frac{1}{k^2}, \quad k \geq 1.$$

Since $\sum_{k=1}^{\infty} \frac{1}{k^2}$ converges, $\sum_{k=1}^{\infty} \frac{k-1}{k^3+3}$ also converges

by the comparison test.

Ex. $\displaystyle\sum_{k=1}^{\infty} \frac{6k^2+1}{2k^3+1}$

Here, when n is large

$$\frac{6k^2+1}{2k^3+1} \approx \frac{6k^2}{2k^3} = \frac{3}{k}$$

and so we expect our series to diverge since $\sum_{k=1}^{\infty} \frac{3}{k}$ does.

Since a fraction decreases if its numerator is made smaller or its denominator is made larger, we have

$$\frac{6k^2+1}{2k^3-1} \geqslant \frac{6k^2}{2k^3-1} \geqslant \frac{6k^2}{2k^3} = \frac{3}{k}, \quad k \geqslant 1$$

so that

$$0 \leq \frac{3}{k} \leq \frac{6k^2+1}{2k^3-1}, \quad k \geqslant 1.$$

Thus since $\sum_{k=1}^{\infty} \frac{3}{k}$ diverges, $\sum_{k=1}^{\infty} \frac{6k^2+1}{2k^3-1}$ also diverges by the comparison test.

The basic ideas behind using the comparison test can be summarized as follows.

1. Look at what happens for large **k** to get an idea of how the terms in the series behave.

2. If you think the series might be convergent, try to find a convergent series whose terms are (eventually) bigger than those of the series you're testing.

   If you think the series might be divergent, try to find a divergent series whose terms are (eventually) smaller (and non-negative) than those of the series you're testing.

3. *Informally*, constant terms in the denominators of the terms can usually be deleted without affecting convergence or divergence.

4. *Informally*, if a polynomial in k appears in either the numerator or denominator of the terms, then the lower powers of k can be deleted without affecting convergence or divergence. This is because the highest power of k dominates when k is large.

WARNING The comparison test only works for series with *non-negative* terms.

Ex. $\displaystyle\sum_{k=3}^{\infty} \frac{k^2-5}{k^3+k+2}$

As $k \to \infty$, k^2 dominates in the numerator and k^3 in the denominator so that

$$\frac{k^2-5}{k^3+k+2} \approx \frac{k^2}{k^3} = \frac{1}{k} \quad \text{for } k \text{ large.}$$

So $\frac{k^2-5}{k^3+k+2}$ behaves like $\frac{1}{k}$ and since $\displaystyle\sum_{k=3}^{\infty} \frac{1}{k}$ diverges, we would expect that

$$\sum_{k=3}^{\infty} \frac{k^2-5}{k^3+k+2} \quad \text{to diverge also.}$$

However, for $k \geq 3$,

$$0 \leq \frac{k^2-5}{k^3+k+2} < \frac{k^2}{k^3} = \frac{1}{k}$$

and the inequality goes in the wrong direction for using the comparison test.

Using algebra, we could find $c > 0$ such that for n large enough

$$\frac{k^2 - 5}{k^3 + k + 2} \geqslant \frac{c}{k} \geqslant 0$$

and since $\sum \frac{c}{k}$ diverges, we could use the comparison test in the usual way to conclude that $\sum_{k=3}^{\infty} \frac{k^2-5}{k^3+k+2}$ diverges.

However, there is an easier way.....

Limit Comparison Test

Suppose $a_k > 0$ and $b_k > 0$ for all k large. If

$$\lim_{n \to \infty} \frac{a_k}{b_k} = c \quad \text{where } c > 0,$$

then the two series $\sum a_k$ $\sum b_k$ either both converge or both diverge.

E.g. in the previous example if we let

$$a_k = \frac{k^2 - 5}{k^3 + k + 2}$$

and $b_k = \frac{1}{k}$, then for $k \geq 3$

$$\frac{a_k}{b_k} = \frac{\dfrac{k^2 - 5}{k^3 + k + 2}}{\dfrac{1}{k}} = \frac{k^3 - 5k}{k^3 + k + 2}$$

and if we divide above and below by k^3 (the highest power of k appearing), we get

$$\frac{a_k}{b_k} = \frac{1 - \frac{5}{k^2}}{1 + \frac{1}{k^2} + \frac{2}{k^2}} \longrightarrow \frac{1}{1} = 1 \quad \text{as } k \to \infty.$$

Thus $\lim_{k \to \infty} \frac{a_k}{b_k} = 1 > 0$ and since

$\sum_{k=3}^{\infty} \frac{1}{k}$ diverges, $\sum_{k=3}^{\infty} \frac{k^2 - 5}{k^3 + k + 2}$ also

diverges by the limit comparison test.

Ex $\displaystyle\sum_{k=1}^{\infty} \frac{k^2+6}{k^4-2k+3}$

For k large

$$a_k = \frac{k^2+6}{k^4-2k+3} \approx \frac{k^2}{k^4} = \frac{1}{k^2}$$

and so we take $b_k = \frac{1}{k^2}$.

Then

$$\frac{a_k}{b_k} = \frac{\frac{k^2+6}{k^4-2k+3}}{\frac{1}{k^2}} = \frac{k^4+6k^2}{k^4-2k+3}$$

$$= \frac{1+\frac{6}{k^2}}{1-\frac{2}{k^3}+\frac{3}{k^4}} \quad \left(\begin{array}{l} \text{divide above and} \\ \text{below by } k^4 \end{array} \right.$$

$$\longrightarrow \frac{1}{1} = 1 \quad \text{as} \quad k \to \infty.$$

Limit comparison test then applies with $c = 1$. Since $\sum_{k=1}^{\infty} \frac{1}{k^2}$ converges, $\sum_{k=1}^{\infty} \frac{k^2+6}{k^4-2k+3}$ also converges. by the limit comparison test.

<u>Ex</u> $\sum_{k=1}^{\infty} \sin\left(\frac{1}{k}\right)$

Since $\sin x \approx x$ for x small (remember that $\lim_{x \to 0} \frac{\sin x}{x} = 1$), this suggests we try the limit comparison test with $a_k = \sin\left(\frac{1}{k}\right)$ and $b_k = \frac{1}{k}$.

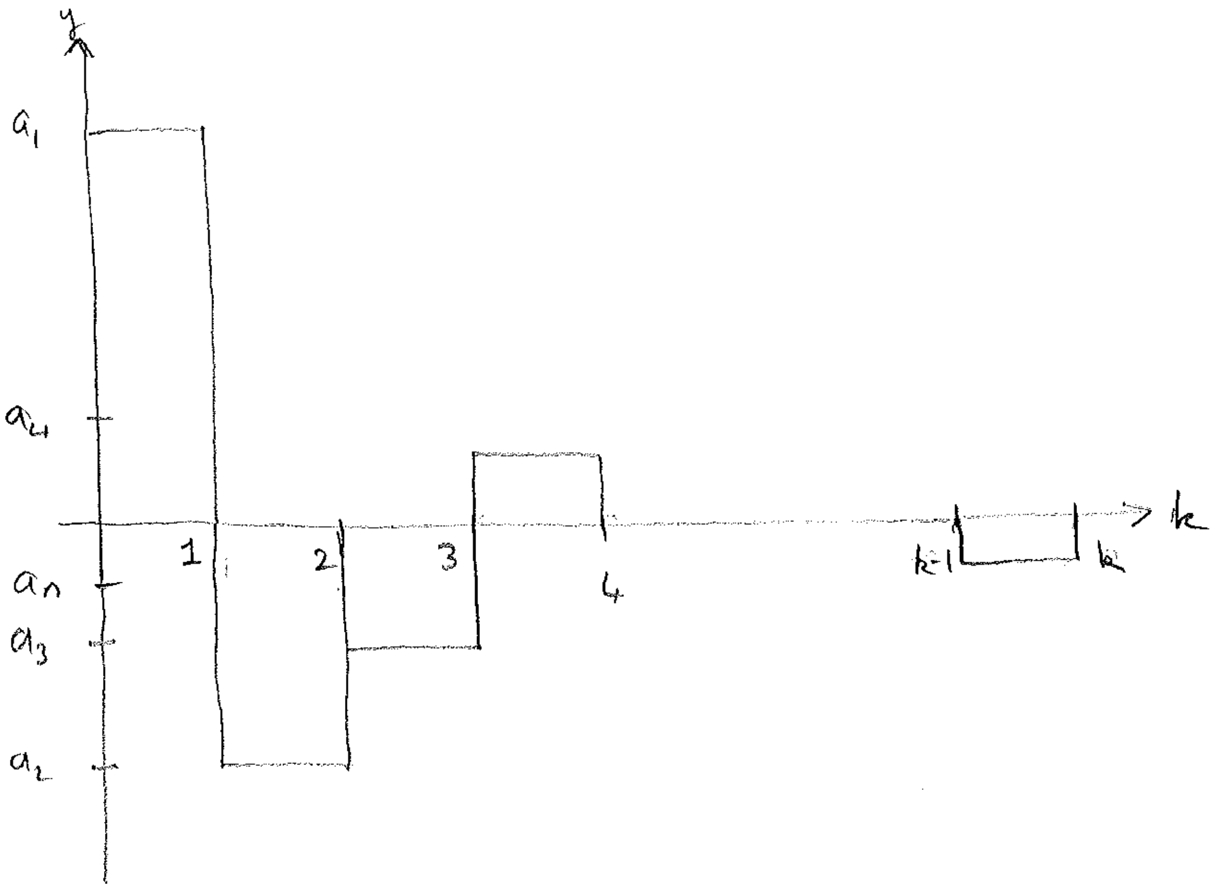
Then

$$\lim_{k \to \infty} \frac{a_k}{b_k} = \lim_{k \to \infty} \frac{\sin\left(\frac{1}{k}\right)}{\frac{1}{k}} = \lim_{x \to 0} \frac{\sin x}{x} = 1.$$

Thus $c = 1$ and since $\sum_{k=1}^{\infty} \frac{1}{k}$ diverges, $\sum_{k=1}^{\infty} \sin\left(\frac{1}{k}\right)$ also diverges by the limit comparison test.

Series with both Positive and Negative Terms

If $\sum a_k$ has both positive and negative terms, then we cannot interpret $\sum a_k$ in terms of area in the same way as before



Note that the area of each rectangle is $|a_n|$, so we can still ask if the sum of the (unsigned) areas $\sum |a_k|$ is finite.

This leads to the following definition.

We say the series $\sum a_k$ is <u>absolutely convergent</u> if the series

$$\sum |a_k|$$

is convergent.

<u>Fact</u> Absolute Convergence Implies Regular Convergence

If $\sum |a_k|$ converges, then $\sum a_k$ converges.

<u>Ex.</u> $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^k}{k^3}$.

Here $a_k = \dfrac{(-1)^k}{k^3}$ and since $|a_k| = \dfrac{1}{k^3}$ and

$\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^3}$ is a convergent p-series $(p = 3 > 1)$,

$\displaystyle\sum_{k=1}^{\infty} \frac{(-1)}{k^3}$ converges also.

Comparison with a Geometric Series

- the Ratio Test

For a geometric series

$$\sum a_k = \sum a x^k$$

the ratio between terms is

$$\frac{a_{k+1}}{a_k} = \frac{a x^{k+1}}{a x^k} = x$$

which is constant.

Recall that if $|x| < 1$, then the series converges and if $|x| \geq 1$, the series diverges.

For many other series $\frac{a_{k+1}}{a_k}$ may not be constant, but we can still say something about convergence.

The Ratio Test

For a series $\sum a_k$ with $a_k > 0$ for k sufficiently large, suppose that the sequence of ratios $\frac{|a_{k+1}|}{|a_k|}$ has a limit

$$\lim_{k \to \infty} \frac{|a_{k+1}|}{|a_k|} = L.$$

i) If $L < 1$, then $\sum a_k$ converges absolutely and is thus convergent.

ii) If $L > 1$ or L is infinite (i.e. $\frac{|a_{k+1}|}{|a_k|}$ grows without limit), then $\sum a_k$ diverges.

iii) If $L = 1$, the test tells us nothing about the convergence or divergence of $\sum a_k$.

Idea of Proof

Convergent case $\lim_{k \to \infty} \frac{|a_{k+1}|}{|a_k|} = L < 1.$

Let x be a number between L and 1,
ie $L < x < 1$.

Then for all k suff. large, say $k \geq k_0$, we have

$$\frac{|a_{k+1}|}{|a_k|} < x.$$

($\frac{|a_{k+1}|}{|a_k|}$ is getting close to L, so eventually it must come below x and stay below x).

Then $|a_{k_0+1}| < |a_{k_0}| x$

$$|a_{k_0+2}| < |a_{k_0+1}| x < |a_{k_0}| x \cdot x = |a_{k_0}| x^2$$

$$|a_{k_0+3}| < |a_{k_0+2}| x < |a_{k_0}| x^2 \cdot x = |a_{k_0}| x^3$$

$\vdots$

etc.

The general pattern we see here is that for $i \geq 0$,

$$|a_{k_0+i}| \leqslant |a_{k_0}| x^i.$$

Comparison with the convergent geometric series

$$\sum_{k=k_0}^{\infty} |a_k| x^k$$

shows that

$$\sum |a_k|$$

is also convergent. Since the first few terms of a series do not affect the convergence or divergence (Part (c) of Th 9.4.3), it follows that

$$\sum |a_k|$$

is convergent. Thus $\sum a_k$ is absolutely convergent and hence convergent.

Divergent Case $\lim_{k\to\infty} \frac{|a_{k+1}|}{|a_k|} = L > 1$ or ∞.

In this case, if we pick x such that

$$L > x > 1,$$

then for sufficiently large k, say $k \geq k_0$

$$|a_{k_0+1}| \geq |a_{k_0}| x > |a_{k_0}|.$$

Thus the sequence $|a_k|$ is (eventually) increasing and so the terms a_k cannot converge to 0. By the divergence test

$\sum a_k$ must diverge.

Ex $\sum_{k=1}^{\infty} \frac{1}{k!}$

Here $k! = k(k-1)(k-2)\cdots 3\cdot 2\cdot 1$, the product of the first k natural numbers.

Here $a_k = \frac{1}{k!}$ and

$$\frac{|a_{k+1}|}{|a_k|} = \frac{\frac{1}{(k+1)!}}{\frac{1}{k!}} = \frac{k!}{(k+1)!}$$

$$= \frac{\cancel{k}\,(\cancel{k-1})\cdots \cancel{2}\cdot\cancel{1}}{(k+1)\cancel{k}\,(\cancel{k-1})\cdots\cancel{2}\cdot\cancel{1}}$$

$$= \frac{1}{k+1} \quad \text{(all other terms cancel !)}$$

Thus $\lim_{k\to\infty} \frac{|a_{k+1}|}{|a_k|} = \lim_{k\to\infty} \frac{1}{k+1} = 0$

and so $\sum_{k=1}^{\infty} \frac{1}{k!}$ converges absolutely and hence converges by the ratio test.

Ex $\displaystyle\sum_{k=1}^{\infty} \frac{(2k)!}{k!\,(k+1)!}$

Here $\displaystyle a_k = \frac{(2k)!}{k!\,(k+1)!}$

So $$\frac{|a_{k+1}|}{|a_k|} = \frac{\dfrac{(2(k+1))!}{(k+1)!\,((k+1)+1)!}}{\dfrac{(2k)!}{k!\,(k+1)!}}$$

$$= \frac{\dfrac{(2k+2)!}{(k+1)!\,(k+2)!}}{\dfrac{(2k)!}{k!\,(k+1)!}}$$

$$= \frac{(2k+2)!\;k!\,(k+1)!}{(2k)!\,(k+1)!\,(k+2)!}$$

$$= \frac{(2k+2)!\;k!}{(2k)!\,(k+2)!}$$

$$= \frac{[(2k+2)(2k+1)(2k)\cdots 2\cdot 1][k\,(k-1)\cdots 2\cdot 1]}{[(2k)(2k-1)\cdots 2\cdot 1][(k+2)(k+1)k\cdots 2\cdot 1]}$$

$$= \frac{(2k+2)(2k+1)}{(k+2)(k+1)}$$

Now divide every term above and below by k (rather like in the limit comparison test) to get

$$\frac{(2+\frac{2}{k})(2+\frac{1}{k})}{(1+\frac{2}{k})(1+\frac{1}{k})} \longrightarrow \frac{2\cdot 2}{1\cdot 1} = 4$$

as $k \to \infty$.

Thus

$$\lim_{k\to\infty} \frac{|a_{k+1}|}{|a_k|} = 4 > 1 \quad \text{and so}$$

$$\sum_{k=1}^{\infty} \frac{(2k)!}{k!\,(k+1)!}$$

diverges by the ratio test.

Ex $\displaystyle\sum_{k=1}^{\infty} \frac{(-2)^k}{k\, 3^k}$

Here $a_k = \dfrac{(-2)^k}{k\,(3)^k}$ and

$$\frac{|a_{k+1}|}{|a_k|} = \frac{\left|\dfrac{(-2)^{k+1}}{(k+1)\,3^{k+1}}\right|}{\left|\dfrac{(-2)^k}{k\,3^k}\right|}$$

$$= \frac{\dfrac{2^{k+1}}{(k+1)\,3^{k+1}}}{\dfrac{2^k}{k\,3^k}}$$

$$= \frac{2^{k+1}\cdot k\,3^k}{2^k\,(k+1)\,3^k}$$

$$= \frac{2\,k}{(k+1)\cdot 3}$$

$$= \frac{2}{3}\cdot\frac{k}{k+1}$$

$$= \frac{2}{3}\cdot\frac{1}{1+\frac{1}{k}} \longrightarrow \frac{2}{3} \text{ as } k\to\infty$$

Thus $\lim\limits_{k \to \infty} \dfrac{|a_{k+1}|}{|a_k|} = \dfrac{2}{3} < 1$ and

so the series $\displaystyle\sum_{k=1}^{\infty} \frac{(-2)^k}{k\, 3^k}$ converges (absolutely)

by the ratio test.

Ex. a) $\sum_{k=1}^{\infty} \frac{1}{k}$ — harmonic series

Here $\frac{|a_{k+1}|}{|a_k|} = \frac{\frac{1}{k+1}}{\frac{1}{k}} = \frac{k}{k+1} = \frac{1}{1+\frac{1}{k}} \to 1$ as $k \to \infty$

So $\lim_{k\to\infty} \frac{|a_{k+1}|}{|a_k|} = 1$ in this case and we already know this series diverges from the last section.

b) $\sum_{k=1}^{\infty} \frac{1}{k^2}$ — p series $(p=2)$

Here $\frac{|a_{k+1}|}{|a_k|} = \frac{\frac{1}{(k+1)^2}}{\frac{1}{k^2}} = \frac{k^2}{(k+1)^2} = \frac{1}{(1+\frac{1}{k})^2} \longrightarrow 1$ as $k \to \infty$.

So $\lim_{k\to\infty} \frac{|a_{k+1}|}{|a_k|} = 1$ and we know this series is convergent $(p>1)$ from the last section.

Taken together, these two examples show why the ratio test tells us nothing when $\lim_{k\to\infty} \frac{|a_{k+1}|}{|a_k|} = 1$

The root test is sometimes useful when it is difficult or impossible to calculate the limit of ratios of terms needed for the ratio test.

The Root Test

Let $\sum a_k$ be a series and suppose that the limit

$$\rho = \lim_{k \to \infty} \sqrt[k]{|a_k|} = \lim_{k \to \infty} (|a_k|)^{\frac{1}{k}}$$

exists.

a) If $\rho < 1$, the series $\sum a_k$ converges absolutely

b) If $\rho > 1$, the series $\sum a_k$ diverges

c) If $\rho = 1$, the series $\sum a_k$ may converge or diverge (both can occur) and the test tells us nothing.

Proof of the test is rather similar to that of the ratio test. It involves comparison with suitable geometric series

Ex. $\displaystyle\sum_{k=2}^{\infty} \left(\frac{4k-5}{2k+1}\right)^{2k}$

Here $$|a_k|^{\frac{1}{k}} = \sqrt[k]{\left(\frac{4k-5}{2k+1}\right)^{2k}}$$

$$= \left(\frac{4k-5}{2k+1}\right)^2$$

$$= \left(\frac{4-5/k}{2+\frac{1}{k}}\right)^2$$

$$\to \left(\frac{4}{2}\right)^2 = 4 \quad \text{as } k \to \infty.$$

Hence this series diverges by the root test.

Ex $$\sum_{k=1}^{\infty} \frac{1}{(\ln(k+1))^k}$$

Here $$\sqrt[k]{|a_k|} = \sqrt[k]{\frac{1}{(\ln(k+1))^k}}$$

$$= \frac{1}{\sqrt[k]{(\ln(k+1))^k}}$$

$$= \frac{1}{\ln(k+1)}$$

$$\to 0 \quad \text{as} \quad k \to \infty.$$

Hence this series converges (absolutely) by the root test.