§9.4 Convergence Tests

Given an infinite series $\sum_{n=1}^{\infty} a_k$, there are several ways to test whether this series diverges or converges.

The simplest of these is the divergence test.

Divergence Test

For an infinite series $\sum_{k=1}^{\infty} a_k$

i) If $\sum_{k=1}^{\infty} a_k$ converges then the individual terms $a_k \to 0$ as $k \to \infty$.

ii) If the individual terms a_k do not converge to 0 as $k \to \infty$, then

$$\sum_{k=1}^{\infty} a_k$$

diverges.

Note that i) and ii) are <u>logically</u> <u>equivalent</u> (i.e. they are either both true or both false). This is an example of the <u>contrapositive</u> in logic.

<u>Ex</u>. $\displaystyle\sum_{k=1}^{\infty} \frac{k}{k+1}$

This series diverges because

$$\frac{k}{k+1} = \frac{1}{1+\frac{1}{k}} \longrightarrow 1 \quad \text{as} \quad k \to \infty.$$

The terms of the series do not converge to 0 as $k \to \infty$ and so the series diverges by the divergence test.

Note that if the terms $a_k \to 0$ as $k \to \infty$, this test tells us nothing.

E.g.

$$\sum_{k=1}^{\infty} \frac{1}{k(k+1)}$$

$$\sum_{k=1}^{\infty} \frac{1}{k}$$

both have terms which converge to 0, but as we saw earlier, the first of these series converges while the second diverges.

Combining Infinite Series

Theorem

a) If $\sum_{k=1}^{\infty} a_k$, $\sum_{k=1}^{\infty} b_k$ both converge, then

so do $\sum_{k=1}^{\infty} (a_k + b_k)$ and $\sum_{k=1}^{\infty} (a_k - b_k)$.

In this case.

$$\sum_{k=1}^{\infty} (a_k + b_k) = \sum_{k=1}^{\infty} a_k + \sum_{k=1}^{\infty} b_k$$

$$\sum_{k=1}^{\infty} (a_k - b_k) = \sum_{k=1}^{\infty} a_k - \sum_{k=1}^{\infty} b_k$$

b) If c is a non-zero constant, then $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} c a_k$ either both converge or both diverge. In the case of convergence

$$\sum_{k=1}^{\infty} c a_k = c \sum_{k=1}^{\infty} a_k.$$

c) Convergence or divergence is unaffected by changing, adding or deleting a finite number of terms in a series. In particular, for any $K \in \mathbb{N}$, the series

$$\sum_{k=1}^{\infty} a_k = a_1 + a_2 + a_3 + \cdots$$

$$\sum_{k=K}^{\infty} a_k = a_K + a_{K+1} + a_{K+2}$$

either both converge or both diverge.

<u>Ex 2</u>. $$\sum_{k=1}^{\infty} \frac{3}{4^k} = \frac{3}{4} + \frac{3}{4^2} + \frac{3}{4^3} + \cdots$$

is a conv. geom series $\left(a = \frac{3}{4}, r = \frac{1}{4}\right)$ with sum

$$\frac{3}{4} \Big/ 1 - \frac{1}{4} = 1$$

while $$\sum_{k=1}^{\infty} \frac{2}{5^{k-1}} = 2 + \frac{2}{5} + \frac{2}{5^2} + \cdots$$

is another conv. geom series $\left(a=2,\ r=\frac{1}{5}\right)$
with sum

$$\frac{2}{1-\frac{1}{4}} = \frac{2}{4/5} = 5/2\,;$$

Hence $$\sum_{k=1}^{\infty}\left(\frac{3}{4^k} - \frac{2}{5^{k-1}}\right)$$

is also a conv. series and

$$\sum_{k=1}^{\infty}\left(\frac{3}{4^k} - \frac{2}{5^{k-1}}\right) = \sum_{k=1}^{\infty}\frac{3}{4^k} - \sum_{k=1}^{\infty}\frac{2}{5^{k-1}}$$

$$= 1 - 5/2$$

$$= -\frac{3}{2}.$$

Ex. $\sum_{n=1}^{\infty}(1-e^{-n})$

$$1-e^{-n} \longrightarrow 1-0 = 1 \quad \text{as} \quad n \to \infty.$$

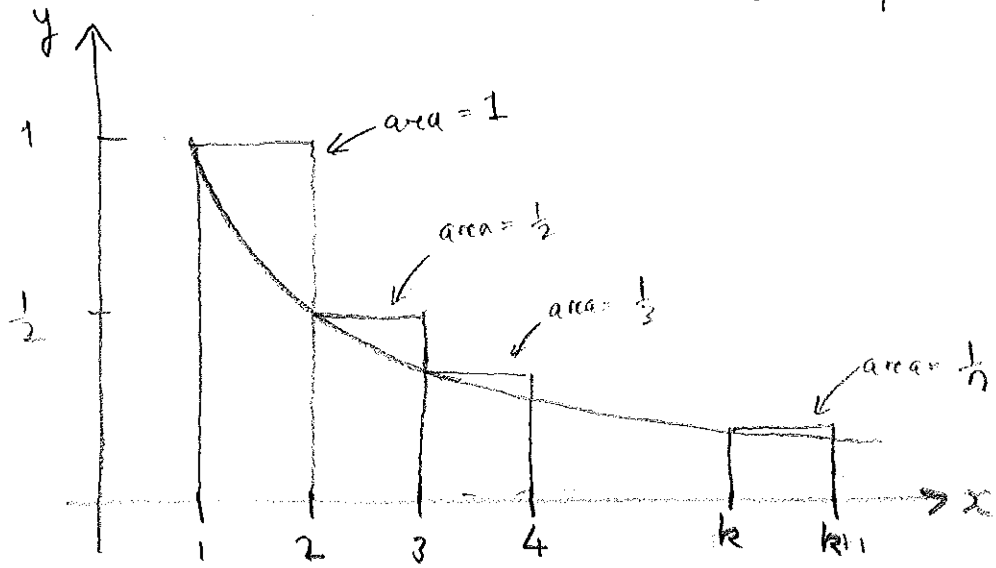
Since the limit of the terms is not 0, this series must diverge by Property 3 above.

Ex. The Harmonic Series

$$\sum_{k=1}^{\infty} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n} + \cdots$$

Claim this series diverges.

See this by approximating $\int_1^\infty \frac{1}{x}\,dx$ as a left-hand sum.



(Recall that $\int_1^\infty \frac{1}{x}\,dx$ diverges.)

Since $\frac{1}{x}$ is decreasing, we see by looking at areas that

$$S_n = 1 + \tfrac{1}{2} + \tfrac{1}{3} + \cdots + \tfrac{1}{n} > \int_1^{n+1} \tfrac{1}{x}\,dx = \Big[\ln x\Big]_1^{n+1}$$

$$= \ln(n+1) - 0.$$

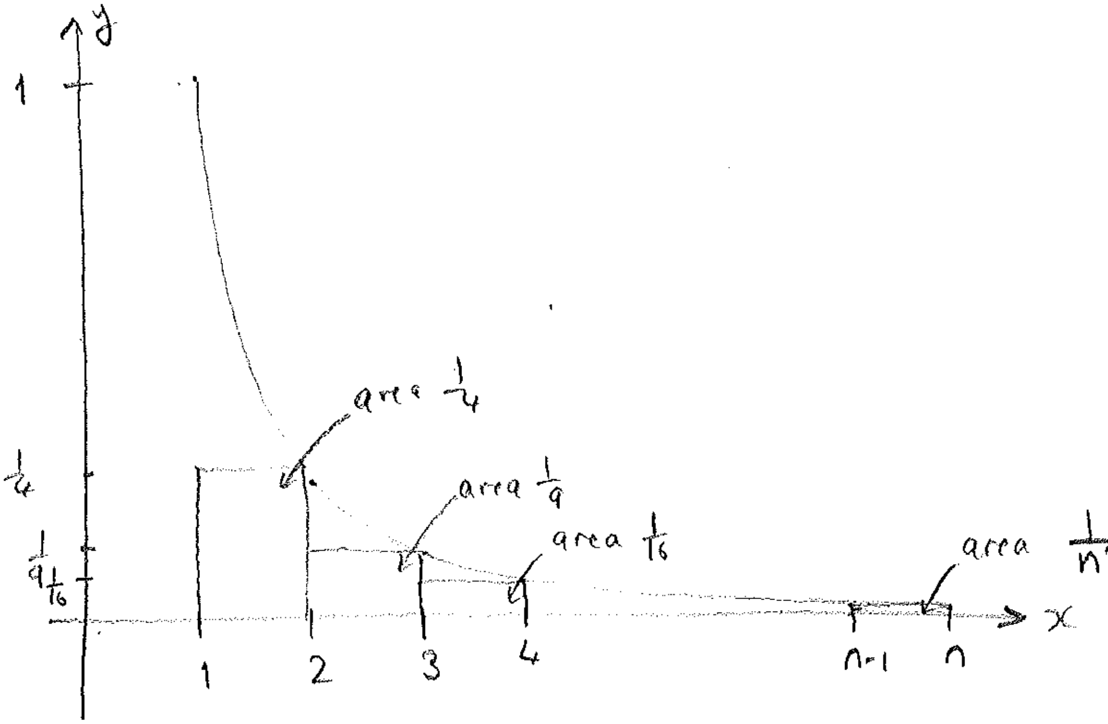
Since $\ln(n+1) \to \infty$ as $n \to \infty$, $S_n \to \infty$ as $n \to \infty$ and so $\sum_{k=1}^{\infty} \frac{1}{k}$ does indeed diverge.

N.b. We already saw a more elementary way of doing this which doesn't use integration.

Ex. $\sum_{k=1}^{\infty} \frac{1}{k^2}$.

In this case, we compare with $\int_1^{\infty} \frac{1}{x^2}\,dx$ and since this integral converges, we guess that $\sum_{k=1}^{\infty} \frac{1}{k^2}$ should also converge.

In this case we should use a right-hand sum.



Again by area we see

$$\frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots + \frac{1}{n^2} \leq \int_1^n \frac{1}{x^2}\,dx$$

$$= \left[-\frac{1}{x}\right]_1^n$$

$$= -\frac{1}{n} - (-1).$$

Thus

$$\frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots + \frac{1}{n^2} \leq 1 - \frac{1}{n}$$

and so, adding 1 to both sides

$$S_n = 1 + \frac{1}{4} + \frac{1}{9} + \qquad + \frac{1}{n^2} \leq 2 - \frac{1}{n}$$

$$< 2.$$

The sequence $\{S_n\}_{n=1}^{\infty}$ is increasing as we're always adding positive terms $\frac{1}{n^2}$ and we've just shown that is bounded above and hence bounded.

By our earlier results on sequences, we can then say that $\{S_n\}_{n=1}^{\infty}$ converges and hence so does

$$\sum_{k=1}^{\infty} \frac{1}{k^2}.$$

In fact, Euler showed that

$$\sum_{n=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}.$$

<u>Remark</u> One can also show $\sum_{n=1}^{\infty} \frac{1}{k^4} = \frac{\pi^4}{90}$, but what about $\sum_{n=1}^{\infty} \frac{1}{k^3}$? If you can do this one, you'll probably get a Fields medal!

The method of the last two examples can be used to prove the following:

The Integral Test

Suppose $a_k = f(k)$, where $f(x)$ is decreasing, cts and positive for $x \geqslant c$.

i) If $\int_c^\infty f(x)\,dx$ converges, then $\sum a_k$ converges.

ii) If $\int_c^\infty f(x)\,dx$ diverges, then $\sum a_k$ diverges.

Recall that we showed in § 7.7 that

$$\int_1^\infty \frac{1}{x^p}\,dx$$

was convergent for $p > 1$ and divergent for $p \geqslant 1$.

Now let us look at the series $\sum_{n=1}^{\infty} \frac{1}{k^p}$ (p-series).

First, if $p \leqslant 0$, $\frac{1}{k^p} = \quad {}^{-p}$ does not tend to 0 as $n \to \infty$. Hence in this case

$$\sum_{n=1}^{\infty} \frac{1}{k^p} \text{ diverges by the divergence test (Property 3).}$$

On the other hand, if $p > 0$, then $\frac{1}{x^p}$ is a positive decreasing function for $x \geqslant 1$ and we can apply the integral test.

Hence, in this case, we can say that

$$\sum_{k=1}^{\infty} \frac{1}{k^p} \text{ diverges for } 0 < p < 1 \text{ and converges for } p > 1.$$

We can summarize what we have found as follows:

The p-series

$$\sum_{k=1}^{\infty} \frac{1}{k^p}$$

<u>converges</u> if $p > 1$ and

<u>diverges</u> if $p \leq 1$.