§ 9.3 Infinite Series

A series is a sum of terms.
Series can be either

- finite : $\sum_{k=1}^{n} a_k := a_1 + a_2 + \cdots + a_n$
- infinite : $\sum_{k=1}^{\infty} a_k := a_1 + a_2 + \cdots + a_n + \cdots$

Examples

$$\sum_{k=1}^{n} k = 1 + 2 + \cdots + n \qquad \left(= \frac{n(n+1)}{2}\right)$$

$$\sum_{k=1}^{\infty} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$$

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots \qquad \left(= \frac{\pi^2}{6} \,!\right)$$

A <u>geometric series</u> is a series (finite or infinite) where the ratio between successive terms is fixed.

e.g.

$$\sum_{k=0}^{5} 3(2)^k = 3 + 6 + 12 + 24 + 48 + 96$$

$$\sum_{k=0}^{\infty} \left(-\frac{1}{3}\right)^k = 1 - \frac{1}{3} + \frac{1}{9} - \frac{1}{27} + \frac{1}{81} - \frac{1}{243} + \frac{1}{729} - \cdots$$

In general

a finite geometric series has the form

$$\sum_{k=0}^{n-1} ax^k = a + ax + ax^2 + \cdots + ax^{n-1}$$

while an infinite geometric series has the form

$$\sum_{k=0}^{\infty} ax^k = a + ax + ax^2 + \cdots + ax^{n-1} + ax^n + \cdots$$

Summing a Finite Geometric Series

Let S_n denote the sum

$$S_n = a + ax + \cdots + ax^n = \sum_{k=0}^{n} ax^k \quad (n \text{ terms}).$$

Then

$$x S_n = ax + ax^2 + \cdots + ax^{n+1}$$

And so

$$S_n - x S_n = a + ax + \cdots + ax^n$$
$$- ax - ax^2 \quad - ax^n - ax^{n+1}$$

$$= a - ax^{n+1}$$ (all terms cancel except the first and the last)

$$= a(1 - x^{n+1})$$

So

$$S_n - x S_n = a(1-x^{n+1})$$

$$S_n(1-x) = a(1-x^{n+1})$$

and if we divide by $1-x$ (which we can do provided $x \neq 1$), we get

$$S_n = a + ax + \cdots + ax^n = a \cdot \frac{1-x^{n+1}}{1-x}, \quad x \neq 1.$$

Note that if $x = 1$, then

$$S_n = \underbrace{a + a + \cdots + a}_{n+1 \text{ terms}} = (n+1)a.$$

Infinite Geometric Series

Recall that the sequence $\{x^n\}_{n=1}^{\infty}$ was convergent to 0 if $|x|<1$ and divergent if $|x| \geqslant 1$.

Thus if $|x|<1$

$$S_n = a\frac{(1-x^{n+1})}{1-x} \longrightarrow \frac{a}{1-x} \quad \text{as} \quad n \to \infty.$$

In this case we say that the infinite geometric series

$$\sum_{k=0}^{\infty} a x^k$$

converges and has sum

$$S = a + ax + ax^2 + \cdots + ax^{n} + \cdots = \frac{a}{1-x}$$

which we can also write as

$$\sum_{k=0}^{\infty} a x^k = \frac{a}{1-x}, \qquad |x|<1.$$

If $|x| > 1$ then provided $a \neq 0$,

$|ax^n| \longrightarrow \infty$ as $n \to \infty$ and so

S_n cannot converge.

If $x = 1$, then provided $a \neq 0$,

$$S_n = a + \cdots + a = na$$

which also doesn't converge.

Finally, if $x = -1$, then provided $a \neq 0$

$$S_n = a - a + a - a + \cdots + (-1)^{n-1}a = \begin{cases} a, & n \text{ odd} \\ 0, & n \text{ even} \end{cases}$$

which also doesn't converge.

Thus, if $|x| \geq 1$, provided $a \neq 0$, the sequence S_n diverges and we say that the infinite series

$$\sum_{n=0}^{\infty} ax^n \text{ diverges.}$$

Ex 1 a) $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots = \sum_{k=0}^{\infty} \left(\frac{1}{2}\right)^k$

Here $a = 1$, $x = \frac{1}{2}$ and $\left|\frac{1}{2}\right| < 1$, so the series is convergent with sum

$$\frac{1}{1 - \frac{1}{2}} = 2.$$

b) $3 - 1 + \frac{1}{3} - \frac{1}{9} + \frac{1}{27} - \cdots = \sum_{k=0}^{\infty} 3\left(-\frac{1}{3}\right)^k$

↑ the first term always gives you a.

dividing the second term by the first gives you x, in this case $x = -1/3$.

Here $a = 3$, $x = -\frac{1}{3}$ and $\left|-\frac{1}{3}\right| < 1$, so the series is convergent with sum

$$\frac{3}{1 - \left(-\frac{1}{3}\right)} = \frac{3}{4/3} = \frac{9}{4}.$$

c) $1 - 7 + 49 - 343 + \cdots = \sum_{k=0}^{\infty} (-7)^k$

Here $a = 1$, $x = -7$ and $|-7| \geqslant 1$, so the series is divergent.

The situation with geometric series is an example of the more general phenomenon of convergence or divergence of a general infinite series.

Let $\sum_{k=1}^{\infty} a_k$ be an infinite series and for each $n \geq 1$ define the nth partial sum

$$S_n = \sum_{k=1}^{n} a_k = a_1 + a_2 + \cdots + a_n$$

The numbers

$$S_1, S_2, S_3, \ldots, S_n, \ldots$$

give us an infinite sequence, the sequence of partial sums $\{S_n\}_{n=1}^{\infty}$.

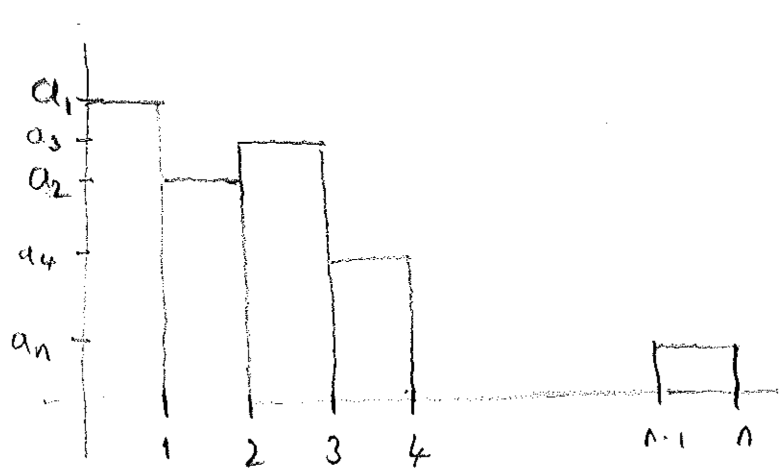
If $\{S_n\}_{n=1}^{\infty}$ is convergent with some (finite) limit S, then we say that $\sum_{n=1}^{\infty} a_n$ is convergent with sum S and we write

$$\sum_{k=1}^{\infty} a_k = S.$$

Otherwise if $\{S_n\}_{n=1}^{\infty}$ is divergent, we say $\sum_{n=1}^{\infty} a_n$ is divergent

Visualizing Series

If we make the following graph where each rectangle over the interval $[n-1, n]$ has height a_n, then $\sum_{k=1}^{\infty} a_k$ represents the sum of all the areas of the rectangles. This is basically an improper integral of type $\int_1^{\infty} f(x)\,dx$.



Ex 2 $$\sum_{k=1}^{\infty} \frac{1}{k(k+1)} = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \cdots$$

Here the nth partial sum is given by

$$S_n = \sum_{k=1}^{n} \frac{1}{k(k+1)} = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \cdots + \frac{1}{n(n+1)}$$

Trick is to use partial fractions on each term

$$\frac{1}{k(k+1)} = \frac{1}{k} - \frac{1}{k+1}$$ (check this!).

Hence

$$S_n = \sum_{k=1}^{n} \left(\frac{1}{k} - \frac{1}{k+1} \right)$$

$$= \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots + \left(\frac{1}{n} - \frac{1}{n+1}\right)$$

$$= 1 + \left(-\frac{1}{2} + \frac{1}{2}\right) + \left(-\frac{1}{3} + \frac{1}{3}\right) + \cdots + \left(-\frac{1}{n} + \frac{1}{n}\right) - \frac{1}{n+1}$$

So nearly everything cancels and

$$S_n = 1 - \frac{1}{n+1} \longrightarrow 1 - 0 = 1$$

as $n \to \infty$.

Hence the sequence $\{S_n\}_{n=1}^{\infty}$ of partial sums is convergent with limit 1 and thus

$$\sum_{k=1}^{\infty} \frac{1}{k(k+1)}$$

converges and has sum 1.

This example is called a <u>telescoping sum</u> (why? Pirates of the Caribbean).

Ex 3 $\sum_{k=1}^{\infty} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$

harmonic series

Let's look at the partial sums S_{2^n} for powers of 2.

$$S_2 = 1 + \frac{1}{2} > \frac{1}{2} + \frac{1}{2} = \frac{2}{2}$$

$$S_4 = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} > S_2 + \left(\frac{1}{4} + \frac{1}{4}\right) > \frac{3}{2}$$

$$S_8 = 1 + \cdots + \frac{1}{8} = S_4 + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right)$$

$$> S_4 + \left(\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\right)$$

$$= S_4 + 4/8$$

$$= S_4 + \frac{1}{2}$$

$$> \frac{3}{2} + \frac{1}{2} = 4/2$$

$$S_{16} = S_8 + \left(\frac{1}{9} + \quad + \frac{1}{16}\right) > S_8 + \left(\frac{1}{16} + \quad + \frac{1}{16}\right)$$

$$= S_8 + \frac{1}{2}$$

$$> \frac{4}{2} + \frac{1}{2} = \frac{5}{2}$$

etc.

Hence $\quad S_{2^n} > \frac{n+1}{2}$.

The numbers S_{2^n} thus grow without limit (ie. $\{S_{2^n}\}_{n=1}^{\infty}$ diverges to ∞).

This means that the whole sequence of partial sums $\{S_n\}_{n=1}^{\infty}$ cannot converge.

Since the sequence of partial sums diverges, it follows that the harmonic series

$$\sum_{k=1}^{\infty} \frac{1}{k} \quad \text{diverges.}$$

(We'll do this again later using the integral test).