§9.10 Finding and Using Taylor Series

In the last section we saw that if we could find all the derivatives $f^{(n)}(x_0)$ of ~~a fn~~ $f(x)$ at $x = x_0$, then we can write down the Taylor series

$$f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \frac{f'''(x_0)}{3!}(x-x_0)^3 + \cdots$$

for f about $x = x_0$.

In practice, even when we know they exist, finding all the derivatives $f^{(n)}(x_0)$ can be very tedious. However, there are some shortcuts which work in certain special cases.

We first list the Taylor series you should know and then describe the tricks you can use to get Taylor series for other fns from them.

Taylor Series You Should Know

1. $$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^k}{k!} + \cdots$$

$$= \sum_{k=0}^{\infty} \frac{x^k}{k!}, \quad \text{valid for every } x$$

2. $$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots + \frac{(-1)^{k+1} x^{2k-1}}{(2k-1)!} + \cdots$$

$$= \sum_{k=1}^{\infty} \frac{(-1)^{k-1} x^{2k-1}}{(2k-1)!}, \quad \text{valid for every } x.$$

3. $$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots + \frac{(-1)^k x^{2k}}{(2k)!} + \cdots$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!}, \quad \text{valid for every } x.$$

4. $$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots + x^k + \cdots$$

$$= \sum_{k=0}^{\infty} x^k, \quad \text{valid for } -1 < x < 1.$$

5. $$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots + (-1)^k x^k + \cdots$$

$$= \sum_{k=0}^{\infty} (-1)^k x^k, \quad \text{valid for } -1 < x < 1$$

6. $$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots + \frac{(-1)^{k-1} x^k}{k} + \cdots$$

$$= \sum_{k=1}^{\infty} \frac{(-1)^{k-1} x^k}{k}, \quad \text{valid for } -1 < x \leq 1$$

7. $$\ln x = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \cdots + \frac{(-1)^{k-1}(x-1)^k}{k} + \cdots$$

$$= \sum_{k=1}^{\infty} \frac{(-1)^{k-1}(x-1)^k}{k}, \quad \text{valid for } 0 < x \leq 2$$

8. $$(1+x)^p = 1 + px + \frac{p(p-1)}{2!} x^2 + \frac{p(p-1)(p-2)}{3!} x^3 + \cdots$$

$$+ \frac{p(p-1)\cdots(p-k+1)}{k!} x^k + \cdots$$

$$= 1 + \sum_{k=1}^{\infty} \frac{p(p-1)\cdots(p-k+1)}{k!} x^k, \quad \text{valid for } -1 < x < 1$$

And now for the tricks......

Tricks for Obtaining New Taylor Series from Old

1. Substitution (composition)

2. Differentiation

3. Integration

4. Multiplication, Division, Composition etc.

With some of these tricks (e.g. 2, 3), we'll be able to find all of the Taylor coefficients at once. Otherwise, we'll only be able to find the first few coefficients, but still more easily than by differentiation.

Ex. Find the Taylor series for e^{-x^2} about $x = 0$.

If we let $f(x) = e^{-x^2}$, then

$$f'(x) = -2x e^{-x^2}$$

$$f''(x) = -2e^{-x^2} + 4x^2 e^{-x^2}$$

etc.

So although, we could find the Taylor coefficients by differentiating, this would quickly get tedious.

However, if we recall that

$$e^y = 1 + y + \frac{y^2}{2!} + \frac{y^3}{3!} + \cdots + \frac{y^k}{k!} + \cdots$$

then if we use substitution and set $y = -x^2$, we get

$$e^{-x^2} = 1 - x^2 + \frac{(-x^2)^2}{2!} + \frac{(-x^2)^3}{3!} + \cdots + \frac{(-x^2)^k}{k!} + \cdots$$

$$= 1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \cdots + \frac{(-1)^k x^{2k}}{k!} + \cdots$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{k!}, \quad \text{valid for every } x.$$

Ex. Find the Taylor series about $x = 0$ for

$$f(x) = \frac{1}{1+x^2}.$$

Recall that we know that

$$\frac{1}{1+y} = 1 - y + y^2 - y^3 + \cdots + (-1)^n y^n + \cdots,$$

$$-1 < y < 1$$

Thus, if we make the substitution $y = x^2$,

$$\frac{1}{1+x^2} = 1 - x^2 + (x^2)^2 - (x^2)^3 + \cdots + (-1)^k (x^2)^k + \cdots$$

$$= \sum_{k=0}^{\infty} (-1)^k x^{2k}$$

This is valid for $-1 < x^2 < 1$

i.e. $0 \le x^2 < 1$

$-1 < x < 1.$

Ex 2. Find the first four terms of the Taylor series for $g(\theta) = e^{\sin\theta}$ about 0.

We know that

$$e^y = 1 + y + \frac{y^2}{2!} + \frac{y^3}{3!} + \cdots + \frac{y^n}{n!} + \cdots$$

$$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots + \frac{(-1)^{n-1}\theta^{2n-1}}{(2n-1)!} + \cdots$$

Thus if we make the substitution $y = \sin\theta$ in the Taylor series for e^y about $y=0$, we get

$$e^{\sin\theta} = 1 + (\theta - \frac{\theta^3}{3!} + \cdots) + \frac{(\theta - \frac{\theta^3}{3!} + \quad)^2}{2!}$$

$$+ \frac{(\theta - \frac{\theta^3}{3!} + \quad)^3}{3!} + \cdots$$

$$= 1 + (\theta - \frac{\theta^3}{3!} + \cdots) + (\frac{\theta^2}{2!} + \qquad)$$

$$+ (\frac{\theta^3}{3!} + \cdots) + \cdots$$

We then rearrange the series to group like powers of θ together (this is allowed).

$$= 1 + \theta + \frac{\theta^2}{2!} + \left(-\frac{\theta^3}{3!} + \frac{\theta^3}{3!}\right) + \cdots$$

$$= 1 + \theta + \frac{\theta^2}{2!} + 0 \cdot \theta^3 + \cdots$$

as desired.

Since the Taylor series for e^y is valid for every value of y and that for $\sin\theta$ is valid for every θ, the (partial) Taylor series for $e^{\sin\theta}$ is valid for every value of θ.

Ex. Find the Taylor series about $x=0$ for $\frac{1}{(1-x)^2}$ from the series for $\frac{1}{1-x}$.

The trick here is to notice that

$$\frac{d}{dx}\left(\frac{1}{1-x}\right) = \frac{1}{(1-x)^2}$$

Now, we already know that

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots + x^k + \cdots,$$

$$-1 < x < 1$$

We then differentiate both sides wrt x, differentiating the series on the right term by term (which is ok) to get

$$\frac{1}{(1-x)^2} = \frac{d}{dx}\left(\frac{1}{1-x}\right) = \frac{d}{dx}\left(1 + x + x^2 + \cdots + x^k + \cdots\right)$$

$$= \frac{d}{dx}(1) + \frac{d}{dx}(x) + \frac{d}{dx}(x^2) + \cdots + \frac{d}{dx}(x^k) + \cdots$$

$$= 0 + 1 + 2x + 3x^2 + 4x^3 + \cdots + kx^{k-1} \cdots$$

Thus

$$\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3 + \cdots, \quad -1 < x < 1$$

$$= \sum_{k=0}^{\infty} (k+1)x^k.$$

<u>Ex</u>. Find the Taylor series for $\arctan x$ about $x = 0$ from that for $\frac{1}{1+x^2}$.

The trick here is to rember that

$$\int \frac{dx}{1+x^2} = \arctan x + C.$$

We already found in an earlier example that

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \cdots, \quad -1 < x < 1.$$

We then take the antiderivative of both sides, integrating the series on the right term by term (which is ok) to get

$$\arctan x = \int \frac{1}{1+x^2}\,dx$$

$$= \int (1 - x^2 + x^4 - x^6 + \cdots)\,dx$$

$$= C + x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots, \quad -1 < x < 1$$

We now just need to find the right value of C. We do this by setting $x = 0$ and remembering that $\arctan 0 = 0$ to get

$$0 = C + 0$$

$$\Rightarrow \quad C = 0$$

Thus

$$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$$

$$= \sum_{k=1}^{\infty} (-1)^{k+1} \frac{x^{2k-1}}{2k-1}, \quad \text{valid for } -1 < x < 1.$$

This last example has the following neat consequence. If we set $x = 1$ and remember that $\arctan 1 = \frac{\pi}{4}$, we get

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$$

so that

$$\pi = 4\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots\right)$$

In principle, we could use this to calculate an approximation to π to any desired accuracy. In practice, this doesn't work so well as the series on the rhs. converges slowly and there are better methods.

Ex. By looking at their Taylor series, decide which of the following fns is largest and which is smallest for x near 0.

a) $1+\sin\theta$, b) e^{θ}, c) $\dfrac{1}{\sqrt{1-2\theta}}$.

a) $\sin\theta$ has Taylor exp. about 0

$$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \cdots$$

So

$$1+\sin\theta = 1 + \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \cdots \quad , \quad \theta \in \mathbb{R}$$

b) e^{θ} has Taylor exp about 0

$$e^{\theta} = 1 + \theta + \frac{\theta^2}{2!} + \frac{\theta^3}{3!} + \frac{\theta^4}{4!} + \cdots$$

Now

$$\frac{1}{\sqrt{1+y}} = (1+y)^{-\frac{1}{2}}$$

and the exp about 0 is

$$(1+y)^{-\frac{1}{2}} = 1 - \frac{y}{2} + \frac{(-\frac{1}{2})(-\frac{3}{2})}{2!} y^2 + \frac{(-\frac{1}{2})(-\frac{3}{2})(-\frac{5}{2})}{3!} y^3 + \cdots$$

$$= 1 - \frac{y}{2} + \frac{3y^2}{8} - \frac{5}{16} y^3 + \cdots, \qquad -1 < y < 1$$

Thus if we substitute $y = -2\theta$

$$\frac{1}{\sqrt{1-2\theta}} = 1 - \frac{(-2\theta)}{2} + \frac{3}{8}(-2\theta)^2 - \frac{5}{16}(-2\theta)^3 + \cdots$$

$$= 1 + \theta + \frac{3}{2}\theta^2 + \frac{5}{2}\theta^3 + \cdots \qquad -\frac{1}{2} < \theta < \frac{1}{2}$$

Let's gather and compare

$$1 + \sin\theta = 1 + \theta - \frac{\theta^3}{3!} + \cdots$$

$$e^\theta = 1 + \theta + \frac{\theta^2}{2} + \frac{\theta^3}{3!} + \cdots$$

$$\frac{1}{\sqrt{1-2\theta}} = 1 + \theta + \frac{3\theta^2}{2} + \frac{5\theta^3}{2} + \cdots$$

Note that all 3 series have the same coefficients for $\underset{\substack{\shortparallel \\ 1}}{\theta^0}$ and θ.

Thus to compare them for θ small, we need to go to the next power of θ, ie θ^2.

For a) the coeff of θ^2 is 0

b) ... $\frac{1}{2}$

c) $\frac{3}{2}$.

Since $0 < \frac{1}{2} < \frac{3}{2}$, for θ near 0 (but $\neq 0$), we have.

$$1 + \sin\theta < e^{\theta} < \frac{1}{\sqrt{1-2\theta}}.$$

Ex Approximate the Integral

$$\int_0^1 e^{-x^2} dx$$

to an accuracy of at least 0.0005 using a McLaurin series for the integrand and by integrating term by term

Already know from before that

$$e^z = 1 + z + \frac{z^2}{2!} + \qquad + \frac{z^k}{k!} + \cdots$$

and substituting $z = -x^2$ gives

$$e^{-x^2} = 1 - x^2 + \frac{(-x^2)^2}{2!} - \qquad + \frac{(-x^2)^k}{k!} + \cdots$$

$$= 1 - x^2 + \frac{x^4}{2!} - \qquad + \frac{(-1)^k x^{2k}}{k!} + \cdots$$

This series has infinite radius of convergence and we can integrate term by term to get

$$\int_0^1 e^{-x^2}\,dx = \int_0^1 \left\{ 1 - x^2 + \frac{x^4}{2!} - \cdots + \frac{(-1)^k x^{2k}}{k!} - \cdots \right\} dx$$

$$= \left[x - \frac{x^3}{3} + \frac{x^5}{5(2!)} - \cdots + \frac{(-1)^k x^{2k+1}}{(2k+1)k!} + - \right]_0^1$$

$$= 1 - \frac{1}{3} + \frac{1}{5(2!)} - \cdots + \frac{(-1)^k}{(2k+1)k!} + \cdots$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)k!} \qquad (\text{using } 0! = 1).$$

Note that this series is easily seen to satisfy the hypotheses of the alternating series test and we recall that if the sum is s and the nth partial sum is s_n, then

$$|s - s_n| < |a_{n+1}| = \left| \frac{(-1)^{k+1}}{(2(k+1)+1)(k+1)!} \right|$$

$$= \frac{1}{(2k+3)(k+1)!}$$

Thus, we will have approximated to three decimal places provided

$$\frac{1}{(2n+3)(n+1)!} < .0005 = 5 \times 10^{-4}.$$

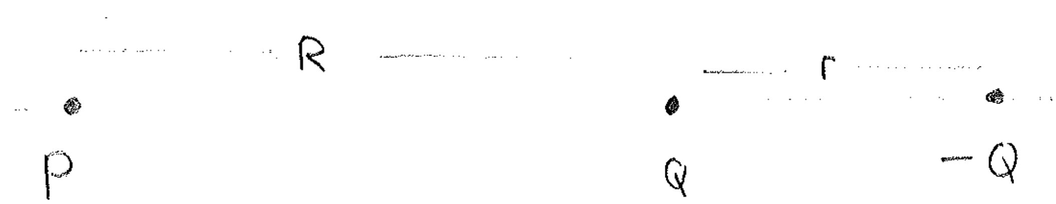
Using a calculator, the smallest value of n for which this happens is $n = 5$. The approximate sum is then.

$$\int_0^1 e^{-x^2} dx \approx 1 - \frac{1}{3} + \frac{1}{5(2!)} - \frac{1}{7(3!)} + \frac{1}{9(4!)} - \frac{1}{11(5!)} \approx .747$$

Actual answer is $\approx .746824$ which agrees with our approximation when rounded up to 3 decimal places.

An Example from Physics.

An electric dipole consists of two charges Q and $-Q$ which are a distance r apart.



The electric field at a point P which is at distance R from the charge Q as shown is given by

$$E = \frac{Q}{R^2} - \frac{Q}{(R+r)^2}.$$

Use series to investigate the behaviour of the electric field far away from the dipole. Show that when $R \gg r$, the electric field is approximately proportional to $\frac{1}{R^3}$.

In order to use a series approximation, we need a variable whose value is small.

We know that R is large while r is fixed but arbitrary.

However, we also know that $R \gg r$, so $\frac{r}{R}$ is small. This suggests we use r/R as our variable and we carry out some algebraic manipulation to make this possible.

$$\frac{1}{(R+r)^2} = \frac{1}{R^2(1+r/R)^2} = \frac{1}{R^2}(1+r/R)^{-2}.$$

We now use the binomial series for $(1+x)^p$ with $p=-2$ and $x = r/R$:

$$\frac{1}{R^2}(1+r/R)^{-2} = \frac{1}{R^2}\left(1 - 2\left(\frac{r}{R}\right) + \frac{(-2)(-3)}{2!}\left(\frac{r}{R}\right)^2 + \frac{(-2)(-3)(-4)}{3!}\left(\frac{r}{R}\right)^3 + \cdots\right)$$

$$= \frac{1}{R^2}\left(1 - \frac{2r}{R} + 3\frac{r^2}{R^2} - 4\frac{r^3}{R^3} + \cdots\right)$$

If we now substitute this into the expression for the electric field, E, we get.

$$E = \frac{Q}{R^2} - \frac{Q}{(R+r)^2}$$

$$= Q\left[\frac{1}{R^2} - \frac{1}{R^2}\left(1 - \frac{2r}{R} + \frac{3r^2}{R^2} - \frac{4r^3}{R^3} + \cdots\right)\right]$$

$$= \frac{Q}{R^2}\left(\frac{2r}{R} - \frac{3r^2}{R^2} + \frac{4r^3}{R^3} - \cdots\right)$$

Note that this binomial expansion will converge since $\left|\frac{r}{R}\right| << 1$ (remember that the series expansion for $(1+x)^p$ is valid for $-1 < x < 1$).

For R very large, $\frac{r}{R}$ is very small and the leading term in the series will be large compared to the other terms (and also to their sum) and so

$$E \approx \frac{Q}{R^2} \cdot \frac{2r}{R} = \frac{2Qr}{R^3}$$

Thus for R large, E is approx. proportional to $\frac{1}{R^3}$ as required.