Chapter 9

Sequences and Series

§ 9.1 Infinite Sequences

An infinite sequence (or simply a sequence) is simply an infinite (ordered) list of numbers

$$a_1, a_2, a_3, \ldots\ldots, a_n, \ldots\ldots$$

The individual numbers a_n are called the terms of the sequence (a_1 is the first term, a_2 the second term and so on).

A common way of writing a sequence is

$$\{a_n\}_{n=1}^{\infty}$$

More formally, an infinite sequence is a function

$$f : \mathbb{N} \longrightarrow \mathbb{R}$$
$$f(n) = a_n$$

Examples

1) $a_n = 1$, for every $n \geq 1$

Get $\{1\}_{n=1}^{\infty} = 1, 1, 1, 1, \ldots$

2) $a_n = (-1)^n$, $n \geq 1$

Get $\{(-1)^n\}_{n=1}^{\infty} = -1, 1, -1, 1, -1, \ldots$

3) $a_n = \frac{1}{n}$, $n \geq 1$

Get $\{\frac{1}{n}\}_{n=1}^{\infty} = 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$

The above are all examples of one of the main ways of representing a sequence - using a formula in n.

A common problem is to find a suitable formula when you're given the first several terms of the sequence.

Ex

a) $3, 6, 12, 24, 48, 96, 192, 384, \ldots$

What we notice here first is that each term is double the previous one. This suggests that 2^n should be part of our formula.

Also, the first term is 3 while all the terms are divisible by 3. This suggests there should be a 3 in our formula.

Try $a_n = 3(2)^n$

This fits all the terms we were given and so (as far as we can tell) this is a formula for our sequence.

b) $\frac{7}{2}, \frac{7}{5}, \frac{7}{8}, \frac{7}{11}, \frac{1}{2}, \frac{7}{17}, \ldots\ldots$

Here the $\frac{1}{2}$ term looks like the rest but if we write $\frac{1}{2}$ as $\frac{7}{14}$, we get

$\frac{7}{2}, \frac{7}{5}, \frac{7}{8}, \frac{7}{11}, \frac{7}{14}, \frac{7}{17}, \ldots$

and the pattern is more evident.

The general term should obviously be a fraction whose numerator is 7. Also the denominator clearly increases by 3 as we go from one term to the next while the first term is 2.

Suggests the denominator is of the form

$3n + k$

and since at 1, we must have $3(1) + k = 2$, we get $k = -1$, and so we get $3n - 1$.

A formula which fits our data is then

$$a_n = \frac{7}{3n-1}, \quad n \geq 1.$$

Consider now the sequence

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, \ldots$$

This is the famous Fibonacci Sequence where each term is the sum of the preceding two. More precisely,

$$a_1 = 1, a_2 = 1, \quad a_n = a_{n-1} + a_{n-2}, \quad n \geqslant 3.$$

This is an example of the second main way of defining a sequence which is recursively where the formula for each term (except possibly the first few) involves previous terms. We generally also need to specify the first few terms to get the recursion started (eg. for Fibonacci, we needed to first give s_1 and s_2).

Ex. Find a recursive formula for each of the following.

a) $1, 2, 6, 24, 120, 720, 5040, \ldots$

Here we notice that each term s_n is n times the previous one. — e.g. $6 = 3(2)$, $720 = 6(120)$. A recursive formula is then

$$a_1 = 1, \quad a_n = n a_{n-1}, \qquad n > 1.$$

Note that here a_n is the famous factorial of n, $n!$ where

$$n! = 1 \cdot 2 \cdot 3 \cdot \cdots \cdot (n-1)(n),$$

the product of the first n natural numbers.

b) $1, 3, 7, 15, 31, 63, 127, \ldots$

Here the first term is 1 while each term is obtained from the last one by doubling and adding 1 (e.g. $7 = 2(3)+1$, $127 = 2(63)+1$).

A recursive defn which fits is then

$$a_1 = 1, \quad a_n = 2a_{n-1} + 1, \quad n > 1.$$

Convergence of Sequences

Consider the sequence

$$\left\{\frac{1}{n}\right\}_{n=1}^{\infty} = 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots\ldots$$

The bigger n is, the smaller $\frac{1}{n}$ becomes and we can clearly make $\frac{1}{n}$ as small as we like by taking n large enough.

This motivates the following defn.

Defn. The sequence $\{a_n\}_{n=1}^{\infty} = a_1, a_2, a_3, \ldots$ has a limit L, written

$$\lim_{n\to\infty} a_n = L$$

if a_n is as close to L as we please whenever n is sufficiently large. If a limit L exists, we say the sequence converges to L. Otherwise, if no limit exists, we say the sequence diverges.

Important Examples

I $\quad x_n = x^n$

If $|x| < 1$, then $\{x^n\}_{n=1}^{\infty}$ converges to 0

e.g if $x = \frac{1}{2}$, we get $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots$

If $|x| > 1$, then $\{x_n\}_{n=1}^{\infty}$ diverges

e.g if $x = -3$, we get $-3, 9, -27, 81, \ldots$

If $x = 1$, then $\{x_n\}_{n=1}^{\infty}$ is just the sequence $1, 1, 1, 1, \ldots$ which (trivially) converges to 1.

Finally, if $x = -1$, then $\{x_n\}_{n=1}^{\infty}$ is the sequence $-1, 1, -1, 1, -1, 1$ which diverges.

II $a_n = \frac{1}{n^p}$ (p fixed).

If $p > 0$, then $\{\frac{1}{n^p}\}_{n=1}^{\infty}$ clearly converges to 0 (e.g. if $p = \frac{1}{2}$, we get

$$1, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{3}}, \frac{1}{2}, \frac{1}{\sqrt{5}}, \ldots$$

If $p < 0$ then $\frac{1}{n^p} = n^{-p}$ gets large as $n \to \infty$ since $-p > 0$. In this case $\{\frac{1}{n^p}\}_{n=1}^{\infty}$ diverges.

Finally, if $p = 0$ $\frac{1}{n^0} = 1$ for every n and so we just have the sequence

$$1, 1, 1, \ldots$$

which is convergent to 1.

Ex.

a) $a_n = (0.8)^n$.

This is of the type in Ex I with $|0.8|<1$ and so converges to 0.

b) $a_n = \dfrac{1-e^{-n}}{1+e^{-n}}$.

Since $|e^{-1}|<1$, $\lim_{n\to\infty} e^{-n} = 0$ and so

$$\lim_{n\to\infty} s_n = \frac{1-0}{1+0} = 1.$$

c) $a_n = \dfrac{1+e^n}{1-e^n}$.

Here we divide above and below by e^n and remember that $\frac{1}{e^n} = e^{-n}$ to get

$$a_n = \frac{e^{-n}+1}{e^{-n}-1}$$

Similarly to b), we then see that $\lim_{n\to\infty} s_n = -1$.

Combining convergent sequences.

<u>Thm</u> Suppose that the sequences $\{a_n\}$ and $\{b_n\}$ converge to Limits L_1 and L_2 respectively and that c is a constant. Then:

a) $\lim_{n \to \infty} c = c$

b) $\lim_{n \to \infty} c a_n = c \lim_{n \to \infty} a_n = c L_1$

c) $\lim_{n \to \infty} (a_n + b_n) = \lim_{n \to \infty} a_n + \lim_{n \to \infty} b_n = L_1 + L_2$

d) $\lim_{n \to \infty} (a_n - b_n) = \lim_{n \to \infty} a_n - \lim_{n \to \infty} b_n = L_1 - L_2$

e) $\lim_{n \to \infty} (a_n b_n) = \lim_{n \to \infty} a_n \cdot \lim_{n \to \infty} b_n = L_1 L_2$

f) $\lim_{n \to \infty} \left(\frac{a_n}{b_n}\right) = \frac{\lim_{n \to \infty} a_n}{\lim_{n \to \infty} b_n} = \frac{L_1}{L_2}$ $\left(\begin{array}{c}\text{provided} \\ L_2 \neq 0\end{array}\right)$.

Ex $\{a_n\}_{n=1}^{\infty} = \left\{\frac{n}{2n+1}\right\}_{n=1}^{\infty}$

Here $a_n = \frac{n}{2n+1}$

and if we divide above and below by n

$$a_n = \frac{1}{2 + \frac{1}{n}}$$

So that $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{2+\frac{1}{n}}$

$$= \frac{\lim_{n\to\infty} 1}{\lim_{n\to\infty} (2 + \frac{1}{n})}$$

$$= \frac{1}{\lim_{n\to\infty} 2 + \lim_{n\to\infty} \frac{1}{n}}$$

$$= \frac{1}{2+0}$$

$$= \frac{1}{2}.$$

Facts about Sequences

A sequence $\{s_n\}_{n=1}^{\infty}$ is called <u>bounded</u> if we can find numbers K and M such that

$$K \leq s_n \leq M \quad \text{for every } n.$$

<u>Fact</u>

A convergent sequence is bounded.

<u>Idea</u>. If $\{s_n\}$ is convergent with limit L, then eventually s_n is close to L (e.g. between $L-1$ and $L-1$) for all sufficiently large n. and so at least these later terms of the sequence are bounded.

The remaining terms are the early ones, but as there are only finitely many of them, they will not affect whether the sequence is bounded.

A sequence $\{s_n\}_{n=1}^{\infty}$ is called <u>monotone</u> if it is either increasing, i.e.

$$s_n < s_{n+1} \quad \forall n \geq 1$$

or it is decreasing

$$s_n > s_{n+1} \quad \forall n \geq 1.$$

<u>Fact.</u>

If $\{s_n\}_{n=1}^{\infty}$ is a sequence which is bounded and monotone, then it is convergent.

<u>Ex.</u> If $s_n = (1 + \frac{1}{n})^n$, one can show that s_n is increasing and that $s_n < 3$ for every n, so that we have a bounded sequence.

By the above fact s_n is then convergent. In fact we already know the limit of this sequence is e using l'Hôpital's rule.

The squeeze theorem.

<u>Thm</u> Let $\{a_n\}, \{b_n\}, \{c_n\}$ be sequences s.t.

$$a_n \le b_n \le c_n \quad \left(\begin{array}{l}\text{for every } n \text{ larger} \\ \text{than some given value } N\end{array}\right.$$

Then, if $\{a_n\}, \{c_n\}$ have the same limit L as $n \to \infty$, $\{b_n\}$ is also convergent with limit L.

<u>Ex</u> $a_n = \dfrac{\sin n}{n}$

Here $-1 \le \sin n \le 1$ for every $n \in \mathbb{N}$

$$\Rightarrow -\frac{1}{n} \le \frac{\sin n}{n} \le \frac{1}{n} \quad \forall n \in \mathbb{N}.$$

Since $\{-\frac{1}{n}\}, \{\frac{1}{n}\}$ are both conv. with the same limit 0, $\{\frac{\sin n}{n}\}$ also convs. to 0 by the squeeze theorem.

A consequence of squeeze

<u>Thm</u> If $\lim_{n\to\infty} |a_n| = 0$, then $\lim_{n\to\infty} a_n = 0$.

<u>Idea</u> $-|a_n| \le a_n \le |a_n| \quad \forall n.$

and $\lim_{n\to\infty} -|a_n| = -\lim_{n\to\infty} |a_n| = 0.$

<u>Ex</u> $a_n = \dfrac{(-1)^n}{\sqrt{n}}$

Here $|a_n| = \dfrac{1}{\sqrt{n}} \to 0$ as $n \to \infty$!

and so $\{a_n\}$ is also conv. to 0.