§ 8.3 Slope Fields, Euler's Method

These are both approximate methods of solving DEs.

Slope Fields

Consider a DE of the form

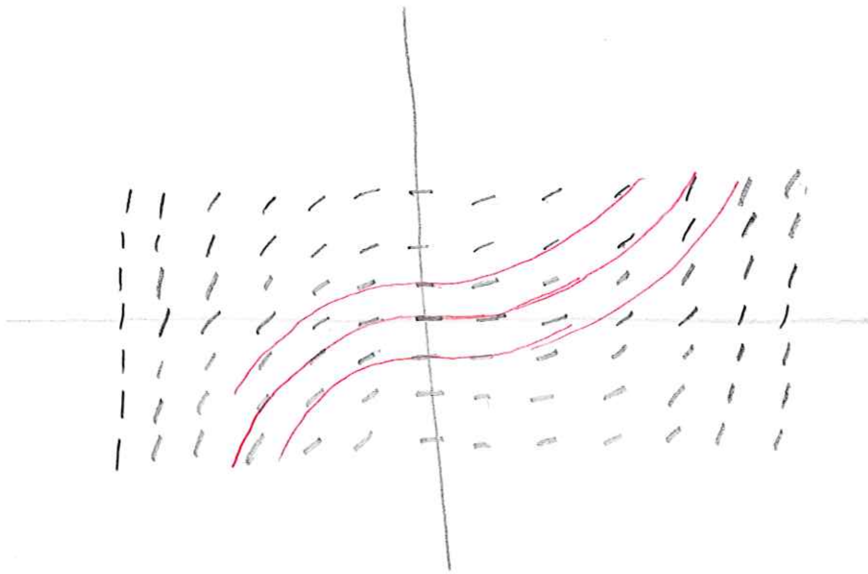
$$y' = f(x,y).$$

At each point (x,y), $f(x,y)$ gives the slope $y' = \frac{dy}{dx}$ of the sol$^{\triangle}$ curve (called an integral curve in the book).

The idea is then to make a short line segment with slope $f(x,y)$ at the point (x,y) for lots of different values of (x,y) and then 'join the lines'.

Ex. $y' = x^2$

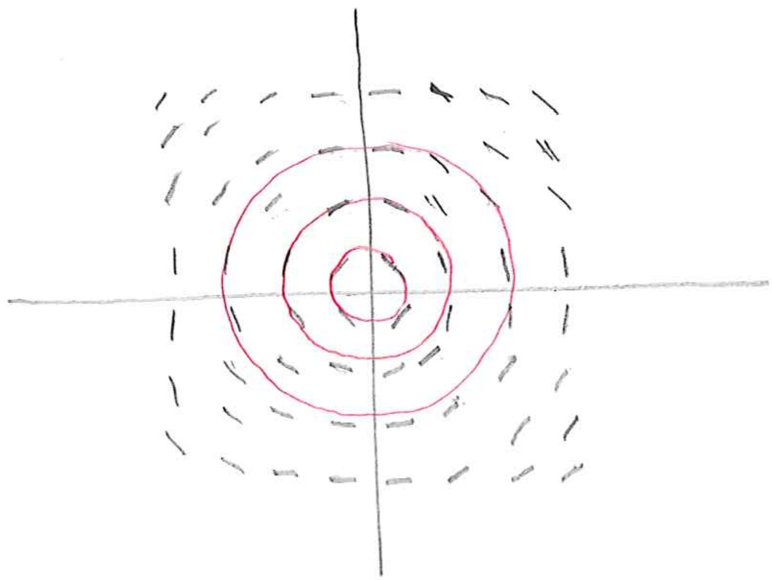
Picture looks something like



Joining the lines by eye, we pick out sol^n curves which look something like the actual sol^{n}s of

$$y = \frac{x^3}{3} + C$$

Ex. $y' = -\frac{x}{y}$



Soln curves appear to be circles about the origin which we can check using separation of variables

$$\frac{dy}{dx} = -\frac{x}{y}$$

$$y\,dy = -x\,dx$$

$$\int y\,dy = -\int x\,dx$$

$$\frac{y^2}{2} = -\frac{x^2}{2} + c'$$

$$\Rightarrow \quad x^2 + y^2 = C \quad , \qquad C = 2c'.$$

Euler's Method

This is a numerical method for finding approximate solutions to a DE.

Idea is to approximate a solution curve by short line segments.

At a point (x,y), the slope of the solution curve to the IVP

$$y' = f(x,y), \quad y(x_0) = y_0$$

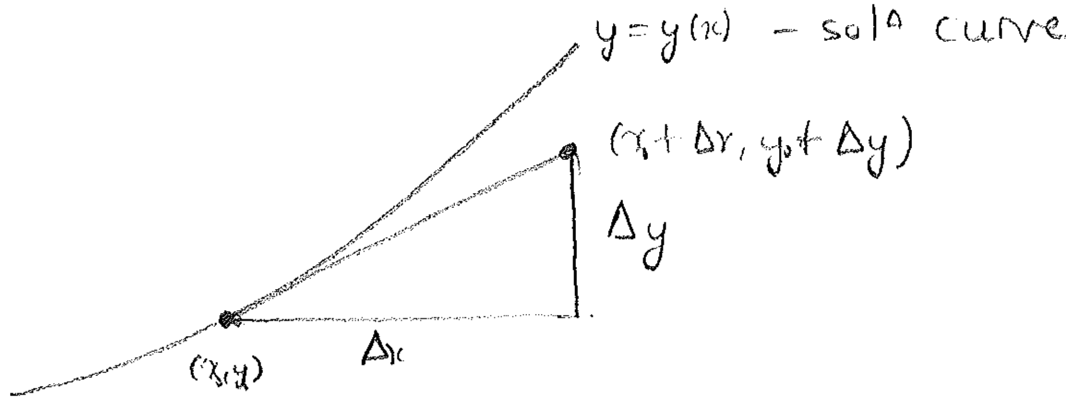
is given by $f(x_0, y_0)$

Thus if Δx is small, if we set

$$\Delta y = f(x_0, y_0)\Delta x,$$

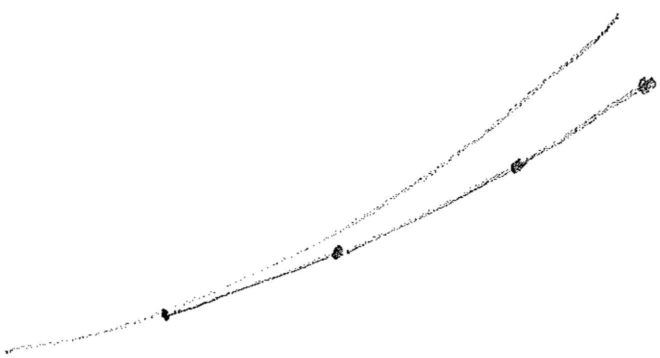
then the point $(x_0 + \Delta x, y_0 + \Delta y)$ is close to the solution curve.

Picture



Note the similarity between this picture and the tangent line approximation and also the slope field method of the last section.

Doing Euler's method repeatedly gives you something like



As you'd expect, the smaller Δx, the better the approximation. Numerical studies show that if Δx is small, then making Δx 10 times smaller makes the error about 10 times smaller.

Euler's Method

To approximate the sol^n of the IVP

$$y' = f(x, y), \quad y(x_0) = y_0$$

proceed as follows;

Step 1 Choose $\Delta x > 0$ to serve as an increment or step size along the x-axis and let

$$x_1 = x_0 + \Delta x, \quad x_2 = x_1 + \Delta x, \quad x_3 = x_2 + \Delta x, \ldots$$

Step 2 Compute successively

$$y_1 = y_0 + f(x_0, y_0)\Delta x$$

$$y_2 = y_1 + f(x_1, y_1)\Delta x$$

$$y_3 = y_2 + f(x_2, y_2)\Delta x$$

$$\vdots$$

$$y_{n+1} = y_n + f(x_n, y_n)\Delta x$$

The numbers y_1, y_2, y_3 in these eq^ns are approximations to $y(x_1), y(x_2), y(x_3), \ldots$