§ 8.2 Separation of Variables

Consider a first order DE of the form

$$h(y)\frac{dy}{dx} = g(x).$$

Such a DE is said to be separable in the sense that if we treat $\frac{dy}{dx}$ like a fraction and multiply both sides by dx, we get

$$h(y)dy = g(x)dx$$

where the lhs depends only on y and the rhs only on x.

If $h(y)$ has antid. $H(y)$ and $g(x)$ has antid. $G(x)$, we can integrate both sides

$$\int h(y)\,dy = \int g(x)\,dx$$

to get

$$H(y) = G(x) + C.$$

To see this formal manipulation actually works, suppose $y = y(x)$ satisfies the above eqn and diff both sides implicitly wrt x

$$\frac{d}{dx}\left(H(y)\right) = \frac{d}{dx}\left(G(x) + C\right)$$

$$H'(y)\frac{dy}{dx} = G'(x)$$

$$\Rightarrow \quad h(y)\frac{dy}{dx} = g(x)$$

(as H is an antid of h and G is an antid of g

So that we do indeed have a solution of the DE.

Summary

To solve a DE of the type

$$h(y)\frac{dy}{dx} = g(x)$$

Step 1 Separate the variables so the eq^{n} is in the form

$$h(y)\,dy = g(x)\,dx$$

Step 2 Integrate both sides (lhs wrt y, rhs wrt x)

$$\int h(y)\,dy = \int g(x)\,dx$$

Step 3 If $H(y)$ is an antid of $h(y)$ and $G(x)$ is an antid. of $g(x)$, then the sol^{n} is described implicitly by

$$H(y) = G(x) + C$$

N.b. sometimes it may not be possible to solve this eq^{n} for y as a fn of x.

Ex 1 Solve the IVP

$$\frac{dy}{dx} = -4xy^2, \qquad y(0) = 1$$

Soln First separate the variables (provided $y \neq 0$)

$$\frac{1}{y^2}\frac{dy}{dx} = -4x$$

$$\frac{1}{y^2}\,dy = -4x\,dx$$

Now integrate both sides

$$\int \frac{1}{y^2}\,dy = \int -4x\,dx$$

Taking antiderivatives gives

$$-\frac{1}{y} = -2x^2 + C$$

$$-y = \frac{1}{-2x^2 + C}$$

$$y = \frac{1}{2x^2 - c}$$

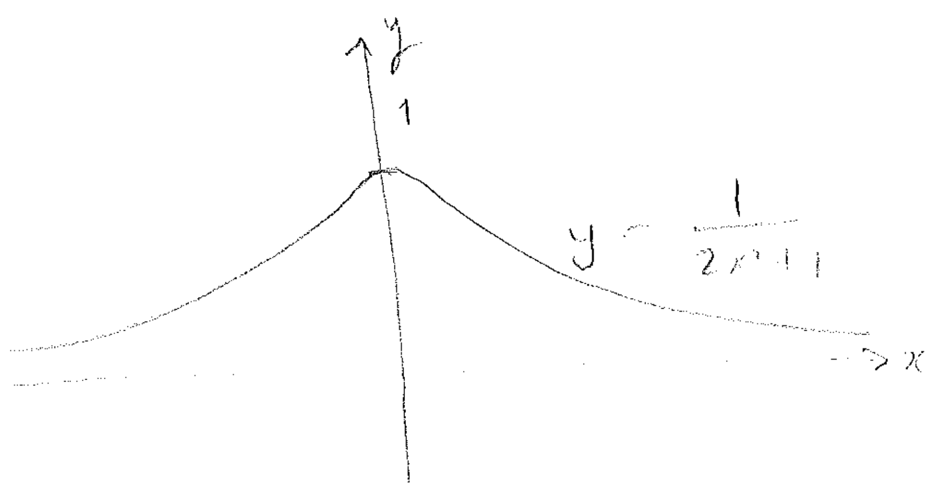
Now apply the IC $y(0) = 1$ i.e.
$y = 1$ when $x = 0$

$$1 = \frac{1}{0^2 - c}$$

$$\Rightarrow \quad c = -1.$$

The solution to the IVP is then

$$y = \frac{1}{2x^2 + 1}.$$



Ex2 Solve the IVP

$$(4y - \cos y)\frac{dy}{dx} - 3x^2 = 0, \quad y(0) = 0$$

Soln First separate the variables

$$(4y - \cos y)\frac{dy}{dx} = 3x^2$$

$$(4y - \cos y)dy = 3x^2 dx$$

Now integrate both sides

$$\int (4y - \cos y)dy = \int 3x^2 dx$$

Taking antids

$$2y^2 - \sin y = x^3 + C$$

Finally, since $y(0) = 0$

$$2(0)^2 - \sin 0 = 0^3 + C$$

Hence $C = 0$ and the desired soln of the IVP is

$$2y^2 - \sin y = x^3$$

We can't solve this to get y as a fn of x, but we can get x as a fn of y

$$x = \sqrt[3]{2y^2 - \sin y}$$

Ex 3 Find a curve in the xy-plane which passes through $(0,3)$ and whose tangent line at a pt. (x,y) has slope $2x/y^2$.

Soln. Since $\frac{dy}{dx}$ is the slope of the tgt line at any given pt, we have the DE

$$\frac{dy}{dx} = \frac{2x}{y^2}$$

while the fact that our curve must pass through $(0,3)$ gives us the IC

$$y(0) = 3.$$

Separating variables gives

$$y^2\,dy = 2x\,dx$$

Integrate both sides

$$\int y^2\,dy = \int 2x\,dx$$

Taking antids.

$$\frac{y^3}{3} = x^2 + C$$

Apply $y(0) = 3$ to find C

$$\frac{3^3}{3} = 0^2 + C$$

$$C = 9.$$

So $\quad \frac{y^3}{3} = x^2 + 9$

which we can solve for y.

$$y^3 = 3x^2 + 27$$

$$y = \sqrt[3]{3x^2 + 27}.$$

Exponential Growth and Decay

Consider the DE

$$\frac{dy}{dx} = ky$$

Separating the variables gives

$$\frac{dy}{y} = k\,dx$$

We then integrate both sides

$$\int \frac{dy}{y} = \int k\,dx$$

$$\ln|y| = kx + C, \quad C \text{ constant.}$$

Taking the exponential of both sides

$$e^{\ln|y|} = e^{kx+C}$$

$$|y| = e^{kx+c}$$

$$|y| = e^{kx} \cdot e^{c}.$$

So $$y = \pm e^{c} e^{kx}$$

and if we let A be the constant $\pm e^{c}$, we get

$$y = A e^{kx}.$$

If in addition, we want our solution to have a particular value y_0 at $x = 0$, we have, on setting $x = 0$,

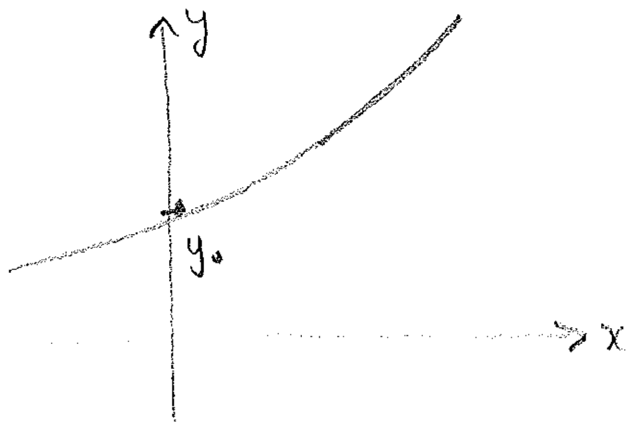
$$y_0 = A e^{k \cdot 0}$$

$$y_0 = A \cdot 1$$

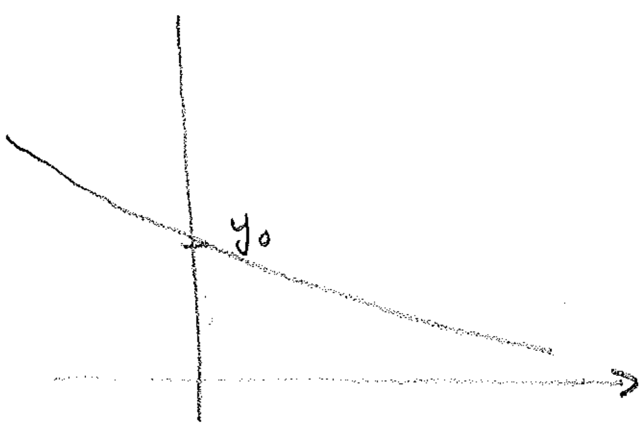
Thus $A = y_0$ and the sol^n of the IVP is

$$y = y_0 e^{kx}.$$

If $k > 0$, the soln grows without bound as $x \to \infty$ and we have <u>exponential growth</u>



If $k < 0$, the soln tends to 0 as $x \to \infty$ and we have <u>exponential decay</u>



Finally, if $k = 0$, then our soln is just the constant fn

$$y = y_0 .$$

Ex. For $k > 0$, find sol^ns of

$$\frac{dH}{dt} = -k(H-20)$$

Separating variables gives

$$\frac{dH}{H-20} = -k\,dt$$

and if we then integrate both sides, we get

$$\int \frac{dH}{H-20} = \int -k\,dt$$

$$\ln|H-20| = -kt + C \quad ; \quad C \text{ constant}$$

$$e^{\ln|H-20|} = e^{-kt+C}$$

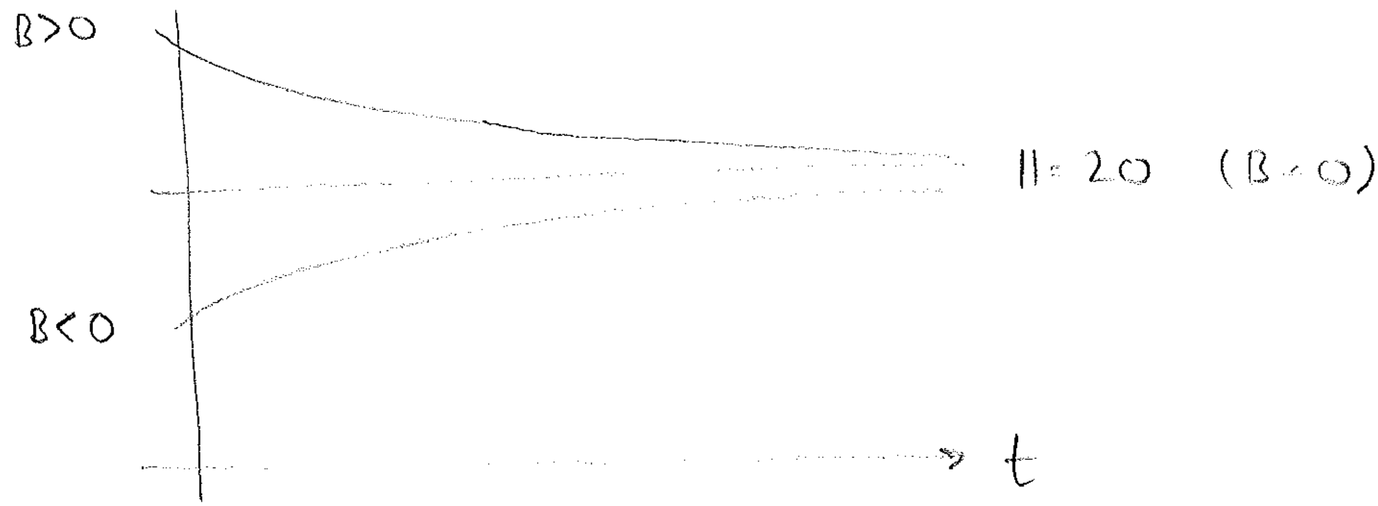
$$|H-20| = e^{C} \cdot e^{-kt}$$

$$H-20 = \pm e^{C} \cdot e^{-kt}$$

$$H - 20 = Be^{-kt} \quad \text{where} \quad B = \pm e^{c}$$

$$H = 20 + Be^{-kt}$$

Solns look like



An eqⁿ of this type could be used to model a situation where an object reaches thermal equilibrium (e.g. a beer left out in the garden when the temp. is 20°C).

Models of Population Growth

We have already seen the exponential model of population growth

$$P = P_0 e^{kt}$$

P_0 is the initial population and k determines the growth rate of the population.

Usually the hard part is to find the value of k.

Ex. The pop. of Mexico in 1980 was about 67 million and in 1984 it was about 75 million. Find an exponential fn which models Mexico's pop. growth and use it to predict when the population is double that in 1980 (ie. 134 million).

Let us measure P in millions of people and t in years since 1980.

$$P = P_0 e^{kt}$$

Here $P_0 = 67$ and so

$$P = 67 e^{kt}$$

In 1984, $t = 4$ and $P = 75$, so

$$75 = 67 e^{4k}$$

$$\frac{75}{67} = e^{4k}$$

Take ln of both sides

$$\ln(75/67) = 4k$$

$$k = \frac{\ln(75/67)}{4} \approx .0282.$$

Thus

$$P = 67 e^{.0282 t}$$

If the pop. is double the amount in 1980, we have $2(67) = 134$ million people.

Set

$$134 = 67 e^{.0282 t}$$

and solve for t.

$$\frac{134}{67} = e^{.0282 t}$$

$$2 = e^{.0282 t}$$

Take ln of both sides.

$$\ln 2 = \ln\left(e^{.0282 t}\right) = .0282 t$$

$$t = \frac{\ln 2}{.0282} \approx 24.57 \text{ years.}$$

Thus the pop. has doubled by around 2004/5.