Chapter 8

Differential Equations

§8.1 Modelling with Differential Equations

A differential equation (DE) in an unknown function $y = y(x)$ is an equation involving derivatives of y (possibly including y itself) and (possibly) x.

E.g.

$$y' = x \sin y$$

$$y'' + y' + 3y = 0$$

$$\sin(y''') + y' = 7\sqrt{4 - x^2}$$

The order of a DE is the power of the highest derivative appearing. The first eqn above is first order, the second second order and the third third order.

A function $y = y(x)$ is a solution of a DE on an open interval I if the eq^{n} is satisfied everywhere on I when we substitute for y and its derivatives.

E.g. $y = e^{2x}$ is a sol^{n} of

$$y' - y = e^{2x}$$

on $(-\infty, \infty)$ since

$$y' = 2e^{2x}, \quad x \in \mathbb{R}$$

and then

$$y' - y = 2e^{2x} - e^{2x} = e^{2x}$$

(which is the rhs.).

However, there are also many other solutions of this DE.

If C is any constant, then

$$y = e^{2x} + Ce^{x}$$ is also a

sol^{n} since now

$$y' = 2e^{2x} + Ce^{x}$$

and $$y' - y = 2e^{2x} + Ce^{x} - (e^{2x} + Ce^{x})$$

$$= e^{2x} \quad (\text{again}).$$

In fact, one can show that all sol^{ns} of this DE are of the form

$$y = e^{2x} + Ce^{x}$$

where C is an arbitrary const.

This is called the <u>general solution</u> of the DE.

Initial Value Problems

To fix the value of the arbitrary constant(s) in the general solution of the DE, we need extra information, namely that the solution (and possibly also some of its derivatives) have specified values at a particular place.

E.g.

$$y(x_0) = y_0$$

means we require our soln $y(x)$ have value y_0 when $x = x_0$.

The condition $y(x_0) = y_0$ is called an Initial Condition (IC) and a differential equation together with an initial condition is called an Initial Value Problem (IVP).

Ex 1 The sol^{n} to the IVP

$$\frac{dy}{dx} - y = e^{2x}, \quad y(0) = 3$$

can be found by substituting $x = 0$, $y = 3$ in the general sol^{n}

$$y = e^{2x} + Ce^{x}$$

we found earlier. Namely

$$3 = e^{2(0)} + Ce^{0}$$

$$3 = 1 + C$$

$$2 = C$$

Hence the sol^{n} of the IVP is

$$y = e^{2x} + 2e^{x}.$$

Uninhibited Population Growth

When resources are unlimited and there are no predators, disease etc., populations (fruit flies, UBI students, wild strawberry plants, etc.) tend to grow at a rate which is proportional to the current population.

If we let $y(t)$ be the pop. at time t, then this behaviour is reflected in the DE for y

$$\frac{dy}{dt} = ky$$

where $k>0$ is a positive constant, called the growth constant. If we require an initial pop. of y_0 individuals, we have the IVP

$$\frac{dy}{dt} = ky, \quad y(0) = y_0.$$

Inhibited Population Growth — Logistic Models.

Here we consider a more realistic model where we have a special number L known as the carrying capacity of our system.

Want the following.

If $y > L$, we have too many individuals so we want $\frac{dy}{dt} < 0$

If $y < L$, we have too few individuals so we want $\frac{dy}{dt} > 0$

If $y = L$, then the population is in a steady state and we want $\frac{dy}{dt} = 0$

Finally, if y is very small ($y \ll L$), we want approximately exponential growth

$$\frac{dy}{dt} \approx ky.$$

All this can be taken into account if $y = y(t)$ satisfies the DE

$$\frac{dy}{dt} = k\left(1 - \frac{y}{L}\right)y$$

If we start with an initial pop. of y_0 then we have the IVP

$$\frac{dy}{dt} = k\left(1 - \frac{y}{L}\right)y, \quad y(0) = y_0.$$

This DE is known as the <u>Logistic Differential Equation</u>

Newton's Law of Cooling

If we place an object in an environment with ambient temperature T_e, it will heat up or cool down at a rate proportional to the temperature difference between the object and the environment. If $T(t)$ is the temp. of the object at time t, then we must have

$$\frac{dT}{dt} = k(T - T_e)$$

for some constant k. Since cold objects will heat up while hot ones will cool down, k must be <u>negative</u>. If we start out at an initial temp T_0, then we have the IVP

$$\frac{dT}{dt} = k(T - T_e), \qquad T(0) = T_0.$$

Hooke's Law for Springs

If a spring is extended a distance x beyond its natural length, then it is subject to a restoring force proportional to that extension

$$F = -kx$$

where k is a constant which must be positive (why?). However using Newton's second law

$$F = ma = m\frac{d^2x}{dt^2},$$

so we obtain the DE

$$m\frac{d^2x}{dt^2} = -kx.$$