§7.8 Improper Integrals

These arise when we try to integrate functions whose graphs are infinite in extent, either horizontally, vertically or both.

An example of a fn whose graph is horizontally infinite is a fn which has a horizontal asymptote, e.g. $\frac{1}{x^2}$, as $x \to \infty$

An example of a fn whose graph is vertically infinite is a fn which has a vertical asymptote, eg. $\frac{1}{x^2}$ again!

but this time as $x \to 0$.

Improper Integrals come in three basic types.

Type I : Horizontally Infinite Region - Infinite Limit(s) of Integration

Ex 1

$$\int_1^\infty \frac{1}{x^2}\,dx$$

Don't know how to handle this (yet!), but we can do

$$\int_1^b \frac{1}{x^2}\,dx$$

and let $b \to \infty$ to see what happens.

$$\int_1^b \frac{1}{x^2}\,dx = \int_1^b x^{-2}\,dx = \left[\frac{x^{-1}}{-1}\right]_1^b$$

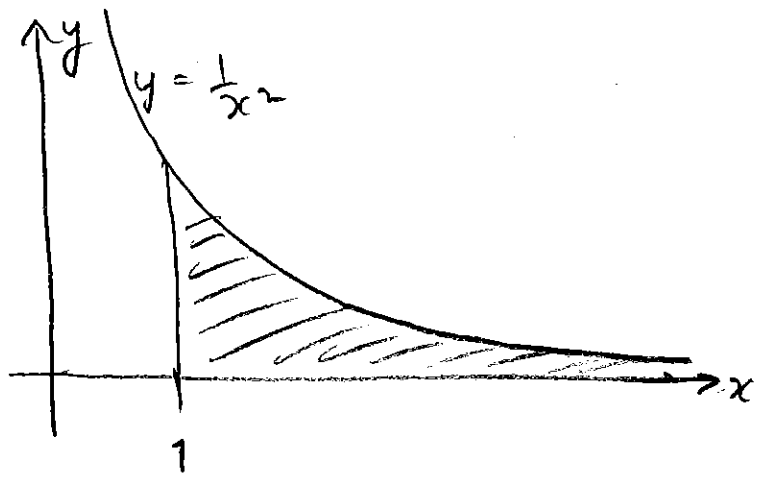
$$= \left[-\frac{1}{x}\right]_1^b$$

$$= -\frac{1}{b} + 1$$

$$\to 1 \text{ as } b \to \infty.$$

We say $\int_1^\infty \frac{1}{x^2}\,dx$ converges and in this sense we can say that $\int_1^\infty \frac{1}{x^2}\,dx = 1$.

If we look at the graph of $\frac{1}{x^2}$



we see that we have a region which is infinite in extent but whose area is still finite!

Defn. Let $f(x)$ be defined for $x \geq a$.

If $\lim_{b \to \infty} \int_a^b f(x)\,dx$ exists and is finite,

we say $\int_a^\infty f(x)\,dx$ converges. In this case we define

$$\int_a^\infty f(x)\,dx := \lim_{b \to \infty} \int_a^b f(x)\,dx$$

Otherwise, we say $\int_a^\infty f(x)\,dx$ diverges.

The convergence / divergence and defn of

$$\int_{-\infty}^b f(x)\,dx$$

are defined similarly.

Ex 2 $\displaystyle\int_1^\infty \frac{1}{\sqrt{x}}\,dx$

$$\int_1^b \frac{1}{\sqrt{x}}\,dx = \int_1^b x^{-\frac{1}{2}}\,dx$$

$$= \left[\frac{x^{\frac{1}{2}}}{\frac{1}{2}}\right]_1^b$$

$$= \left[2\sqrt{x}\right]_1^b$$

$$= 2\sqrt{b} - 2\sqrt{1}$$

$$= 2\sqrt{b} - 2 \longrightarrow \infty \quad \text{as} \quad x \to \infty.$$

Thus $\displaystyle\int_1^\infty \frac{1}{\sqrt{x}}\,dx$ diverges.

Ex.3 $\displaystyle\int_0^\infty e^{-5x}\,dx$

$$\int_0^b e^{-5x}\,dx = \left[-\frac{1}{5}e^{-5x}\right]_0^b$$

$$= -\frac{1}{5}e^{-5b} - \left(-\frac{1}{5}e^{-0}\right)$$

$$= -\frac{1}{5}e^{-5b} + \frac{1}{5}$$

$$\longrightarrow \frac{1}{5} \quad \text{as} \quad b \longrightarrow \infty.$$

Thus $\displaystyle\int_0^\infty e^{-5x}\,dx$ converges and

$$\int_0^\infty e^{-5x}\,dx = \frac{1}{5}.$$

Ex 4 For which values of the exponent p does the integral

$$\int_1^\infty \frac{1}{x^p}\,dx$$

converge, diverge?

Let us first consider the case where $p \neq 1$
(Q: why do we need to look at $p=1$ separately?)

$$\int_1^b \frac{1}{x^p}\,dx = \int_1^p x^{-p}\,dx = \left[\frac{x^{-p+1}}{-p+1}\right]_1^b$$

$$= \frac{1}{1-p}\left(b^{1-p} - 1\right).$$

If $p>1$, $1-p<0$ and $b^{1-p} \longrightarrow 0$ as $b \to \infty$
and in this case

$$\int_1^b \frac{1}{x^p}\,dx \quad \text{converges to} \quad \frac{1}{1-p}(0-1) = \frac{1}{p-1}.$$

and so $\int_1^\infty \frac{1}{x^p}\,dx$ converges and

$$\int_1^\infty \frac{1}{x^p}\,dx = \frac{1}{p-1}.$$

If $p < 1$, $1-p > 0$ and $b^{1-p} \to \infty$ as $b \to \infty$

In this case

$$\int_1^b \frac{1}{x^p}\,dx = \frac{1}{1-p}\left(b^{1-p} - 1\right) \to \infty \quad \text{as } b \to \infty$$

and so $\int_1^\infty \frac{1}{x^p}\,dx$ diverges.

Finally, we consider the case $p = 1$

$$\int_1^b \frac{1}{x}\,dx = \Big[\ln|x|\Big]_1^b = \ln b - \ln 1$$

$$= \ln b \to \infty \quad \text{as } x \to \infty,$$

Thus $\int_1^\infty \frac{1}{x}\,dx$ diverges.

Summary

$$\int_1^\infty \frac{1}{x^p}\,dx \quad \begin{cases} \text{diverges for } p \leq 1 \\ \text{converges and has value } \dfrac{1}{1-p} \text{ for } p > 1. \end{cases}$$

N.b. this example is important!

Ex 5 $\displaystyle\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx.$

Here the region is infinite horizontally in both directions. Idea is to split it up into two separate integrals and investigate the convergence of each separately.

A good place to make the split is $x=0$.

So let's look at $\int_{-\infty}^{0} \frac{1}{1+x^2}\,dx$.

$$\int_{a}^{0} \frac{1}{1+x^2}\,dx = \Big[\arctan x\Big]_{a}^{0}$$

$$= \arctan 0 - \arctan a.$$

$$= 0 - \arctan a.$$

$$\longrightarrow 0 - \left(-\frac{\pi}{2}\right) = \frac{\pi}{2} \text{ as } a \to -\infty.$$

So $\int_{-\infty}^{0} \frac{1}{1+x^2}\,dx$ converges and $\int_{-\infty}^{0} \frac{1}{1+x^2}\,dx = \frac{\pi}{2}$.

A similar argument shows that $\int_{0}^{\infty} \frac{1}{1+x^2}\,dx$ also converges and $\int_{0}^{\infty} \frac{1}{1+x^2}\,dx = \frac{\pi}{2}$.

Thus $\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx$ converges and

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx = \frac{\pi}{2} + \frac{\pi}{2} = \pi.$$

Formally an improper integral over all of $\mathbb{R}$ of the type

$$\int_{-\infty}^{\infty} f(x)\,dx$$

is said to converge if for some (any) c, both

$$\int_{-\infty}^{c} f(x)\,dx, \quad \int_{c}^{\infty} f(x)\,dx$$

converge. In this case, we set

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{c} f(x)\,dx + \int_{c}^{\infty} f(x)\,dx$$

If for some c, either or both of $\int_{-\infty}^{c} f(x)\,dx$, $\int_{c}^{\infty} f(x)\,dx$ diverge, then we say $\int_{-\infty}^{\infty} f(x)\,dx$ diverges.

Quite often we need to use substitution with improper integrals.

<u>Ex 6</u> $\displaystyle\int_{-\infty}^{\infty} x^3 e^{-x^4}\,dx$

Again, we split this up into

$$\int_{-\infty}^{0} x^3 e^{-x^4}\,dx \quad \text{and} \quad \int_{0}^{\infty} x^3 e^{-x^4}\,dx$$

and investigate each of these separately.

For the first integral, we look at

$$\int_{a}^{0} x^3 e^{-x^4}\,dx.$$

e^{-x^4} is a composite fn which suggests we try the substitution

$$u = x^4, \quad du = 4x^3\,dx$$

Limits $\quad x = a, \quad u = a^4$

$x = 0, \quad u = 0$

Rewrite the integral in terms of u to get

$$\int_{a^4}^{0} e^{-u} \cdot \frac{du}{4}.$$

Note that the limits are the wrong way round, which can be taken care of using a minus sign

$$= -\frac{1}{4} \int_0^{a^4} e^{-u}\, du$$

$$= -\frac{1}{4} \left[-e^{-u} \right]_0^{a^4}$$

$$= -\frac{1}{4} \left(-e^{-a^4} - (-e^{0}) \right)$$

$$= \frac{e^{-a^4}}{4} - \frac{1}{4} \longrightarrow \frac{-1}{4} \quad \text{as} \quad a \to -\infty.$$

Hence

$$\int_{-\infty}^{0} x^3 e^{-x^4}\, dx$$

converges and has value $\frac{-1}{4}$.

Similarly

$$\int_{0}^{\infty} x^3 e^{-x^4}\, dx$$

converges and has value $\frac{1}{4}$. (Q. how do I know this without having to think too hard?)

Hence

$$\int_{-\infty}^{\infty} x^3 e^{-x^4}\, dx$$

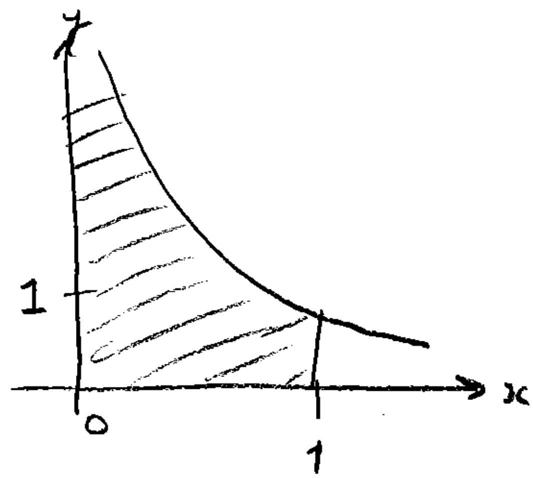
converges and has value $\frac{-1}{4} + \frac{1}{4} = 0$.

Type II Vertically Infinite Region

- Vertical Asymptote(s)

Ex 7.

$$\int_0^1 \frac{1}{\sqrt{x}}\,dx$$



Problem is that the integrand blows up at 0 (vertical asymptote).

Idea is to consider

$$\int_a^1 \frac{1}{\sqrt{x}}\,dx$$

where a is small and positive. Then see what happens as $a \longrightarrow 0_+$ (i.e. tends to 0 from the right).

$$\int_a^1 \frac{1}{\sqrt{x}}\,dx = \int_a^1 x^{-\frac{1}{2}}\,dx$$

$$= \left[2x^{\frac{1}{2}} \right]_a^1$$

$$= 2\sqrt{1} - 2\sqrt{a}$$

$$= 2 - 2\sqrt{a} \longrightarrow 2 \text{ as } a \to 0_+ .$$

We say $\int_0^1 \frac{1}{\sqrt{x}}\,dx$ converges and in this sense we can say that $\int_0^1 \frac{1}{\sqrt{x}}\,dx = 2$.

Defn. Suppose $f(x)$ is defined on $(a, b]$ and that $f(x) \to \infty$ or $f(x) \to -\infty$ as $x \to a_+$.

If $\lim_{c \to a_+} \int_c^b f(x)\,dx$ exists and is finite, we say $\int_a^b f(x)\,dx$ converges. In this case we define

$$\int_a^b f(x)\,dx = \lim_{c \to a_+} \int_c^b f(x)\,dx.$$

Otherwise, we say $\int_a^b f(x)\,dx$ diverges.

Defns are similar if instead $f(x)$ is defined on $[a, b)$ and $f(x) \to \infty$ or $f(x) \to -\infty$ as $x \to b_-$.

<u>Ex 8</u> $\displaystyle\int_0^2 \frac{1}{(x-2)^2}\,dx$

Here the problem is at the upper limit, 2, as

$$\frac{1}{(x-2)^2} \to +\infty \quad \text{as} \quad x \to 2-.$$

$$\int_0^c \frac{1}{(x-2)^2}\,dx = \int_0^c (x-2)^{-2}\,dx$$

$$= \Big[-(x-2)^{-1}\Big]_0^c$$

[If you like, let $w = x-2$]

$$= \Big[-\frac{1}{x-2}\Big]_0^c$$

$$= -\frac{1}{c-2} - \Big(-\frac{1}{-2}\Big)$$

$$= \frac{1}{2-c} - \frac{1}{2} \to +\infty \quad \text{as} \quad c \to 2-.$$

Thus $\displaystyle\int_0^2 \frac{1}{(x-2)^2}\,dx$ diverges.

Ex 9 For which values of p does

$$\int_0^1 \frac{1}{x^p}\,dx$$

converge, diverge?

Again we first consider $p \neq 1$ and then $p = 1$.

When $p \neq 1$

$$\int_c^1 \frac{1}{x^p}\,dx = \int_c^1 x^{-p}\,dx = \left[\frac{x^{-p+1}}{-p+1}\right]_c^1$$

$$= \frac{1}{-p+1} - \frac{c^{-p+1}}{-p+1}$$

$$= \frac{1}{1-p} - \frac{c^{1-p}}{1-p}.$$

If $p < 1$, $1-p > 0$ and

$$\int_c^1 \frac{1}{x^p}\,dx = \frac{1}{1-p} - \frac{c^{1-p}}{1-p} \longrightarrow \frac{1}{1-p} - 0 = \frac{1}{1-p}$$

as $c \longrightarrow 0_+$.

In this case, $\int_0^1 \frac{1}{x^p}\,dx$ converges and

$$\int_0^1 \frac{1}{x^p}\,dx = \frac{1}{1-p}.$$

If $p > 1$, $1-p < 0$ and

$$\int_c^1 \frac{1}{x^p}\,dx = \frac{1}{1-p} - \frac{c^{1-p}}{1-p} = \frac{1}{1-p} + \frac{c^{1-p}}{p-1} \longrightarrow +\infty$$

as $c \to 0_+$,

so the integral diverges.

Finally when $p = 1$

$$\int_c^1 \frac{1}{x}\,dx = \Big[\ln|x|\Big]_c^1 = \ln 1 - \ln c$$

$$= 0 - \ln c$$

$$\longrightarrow \infty \quad \text{as } c \to 0_+.$$

Thus $\int_0^1 \frac{1}{x}\,dx$ diverges.

Summary

$$\int_0^1 \frac{1}{x^p}\,dx \quad \begin{cases} \text{converges if } p<1 \\ \text{and has value } \dfrac{1}{1-p}. \\ \text{diverges if } p \geq 1. \end{cases}$$

N.b. this example is important!

You should compare this with the results for $\int_1^\infty \frac{1}{x^p}\,dx$.

Q. Can use a substitution to get the results for $\int_0^1 \frac{1}{x^p}\,dx$ from those for $\int_1^\infty \frac{1}{x^p}\,dx$ and vice versa?

Ex 10 $\displaystyle\int_1^5 \frac{1}{x\ln x}\,dx$

Here the problem is at $x=1$ (why?), so we need to do

$$\int_a^5 \frac{1}{x\ln x}\,dx$$

and then let $a \to 1_+$.

For this we use the substitution

$$u = \ln x, \qquad du = \frac{dx}{x}$$

Limits When $x = a$, $u = \ln a$
$x = 5$, $u = \ln 5$.

Get $$\int_{\ln a}^{\ln 5} \frac{du}{u} = \Big[\ln|u|\Big]_{\ln a}^{\ln 5}$$

$$= \ln(\ln 5) - \ln(\ln a)$$

as $a \to 1_+$, $\ln a \to 0_+$,

while $\ln(\ln a) \to -\infty$

(show graph if needed).

Hence

$$\lim_{a \to 1_+} \int_a^5 \frac{1}{x \ln x}\, dx$$

does not exist and so

$$\int_1^5 \frac{1}{x \ln x}\, dx$$

diverges.

Ex 11 $\displaystyle\int_{-1}^{1} \frac{1}{x^{2/3}}\,dx$

Problems here on both sides of $x = 0$ where $\frac{1}{x^{2/3}}$ has a vertical asymptote.

Split up the integral into

$$\int_{-1}^{0} \frac{1}{x^{2/3}}\,dx + \int_{0}^{1} \frac{1}{x^{2/3}}\,dx$$

and investigate the convergence of each integral separately.

From the last example

$\displaystyle\int_{0}^{1} \frac{1}{x^{2/3}}\,dx$ converges and has value $\frac{1}{1-2/3} = 3$.

Similarly $\displaystyle\int_{-1}^{0} \frac{1}{x^{2/3}}\,dx$ converges and also has value 3.

Thus $\int_{-1}^{1} \frac{1}{x^{2/3}}\,dx$ converges and

$$\int_{-1}^{1} \frac{1}{x^{2/3}}\,dx = \int_{-1}^{0} \frac{1}{x^{2/3}}\,dx + \int_{0}^{1} \frac{1}{x^{2/3}}\,dx$$

$$= 3 + 3$$

$$= 6.$$

<u>Q.</u> What is wrong with the following?

$$\int_{-1}^{1} \frac{1}{x^2}\,dx = \int_{-1}^{1} x^{-2}\,dx = \left[\frac{x^{-1}}{-1}\right]_{-1}^{1}$$

$$= \left[-\frac{1}{x}\right]_{-1}^{1}$$

$$= -1 - \left(-\frac{1}{-1}\right)$$

$$= -2.$$

A. The integrand needs to be integrable on the interval of integration!

Type III Vertically and Horizontally Infinite Region

This is basically a combination of types I and II which we handle by breaking up the interval of integration.

Ex 12 $\displaystyle\int_0^\infty \frac{1}{x^2}\,dx.$

Break the interval at $x = 1$. and examine $\displaystyle\int_0^1 \frac{1}{x^2}\,dx$ and $\displaystyle\int_1^\infty \frac{1}{x^2}\,dx$ separately.

From what we saw earlier

$\displaystyle\int_0^1 \frac{1}{x^2}\,dx$ diverges and $\displaystyle\int_1^\infty \frac{1}{x^2}$ converges.

Thus $\displaystyle\int_0^\infty \frac{1}{x^2}\,dx$ diverges overall.