§ 7.5 Partial Fractions

This is a method for rewriting rational functions

$$R(x) = \frac{P(x)}{Q(x)}, \quad P, Q \text{ polynomials}$$

in a form which makes them easier to integrate.

Ex 1 Find A, B for which

$$\frac{1}{(x-2)(x-5)} = \frac{A}{x-2} + \frac{B}{x-5}$$

Put the rhs over a common denominator

$$= \frac{A(x-5)}{(x-2)(x-5)} + \frac{B(x-2)}{(x-2)(x-5)}$$

So

$$\frac{1}{(x-2)(x-5)} = \frac{A(x-5) + B(x-2)}{(x-2)(x-5)}$$

Multiply both sides by $(x-2)(x-5)$

$$1 = A(x-5) + B(x-2)$$

$$1 = (A+B)x - 5A - 2B$$

Lhs & rhs are both polys. in x and they are equal iff the coefficients of the different powers of x match.

So

x $\quad A + B = 0 \qquad (1)$

1 $\quad -5A - 2B = 1 \qquad (2)$

(1) $\Rightarrow$ $B = -A$ & if we subst in (2) we get

$$-5A + 2A = 1$$

$$-3A = 1$$

$$A = -\frac{1}{3}$$

$$B = -A = \frac{1}{3}.$$

So $$\frac{1}{(x-2)(x-5)} = -\frac{1}{3(x-2)} + \frac{1}{3(x-5)}$$

Now let's see why this is so useful....

<u>Ex 2</u> $$\int \frac{1}{(x-2)(x-5)}\,dx$$

From above we can rewrite this as

$$\int \left(\frac{-1}{3(x-2)} + \frac{1}{3(x-5)} \right) dx$$

$$= -\frac{1}{3}\int \frac{dx}{x-2} + \frac{1}{3}\int \frac{dx}{x-5}$$

$$= -\frac{1}{3}\ln|x-2| + \frac{1}{3}\ln|x-5| + C$$ by V-2.6 from table.

Ex 3 $\displaystyle\int \frac{x+2}{x^2+x}\,dx$

Factor the denominator x^2+x as $x(x+1)$ and write

$$\frac{x+2}{x^2+x} = \frac{x+2}{x(x+1)} = \frac{A}{x} + \frac{B}{x+1}$$

$$= \frac{A(x+1)+Bx}{x(x+1)}$$

So $\quad x+2 = A(x+1) + Bx$

$\qquad\qquad = (A+B)x + A$

Thus

x / $\quad A + B = 1 \qquad (1)$

1 / $\quad A = 2 \qquad (2)$

Sub for A in (1)

$2 + B = 1$

$B = -1$

Thus

$$\frac{x+2}{x^2+x} = \frac{2}{x} - \frac{1}{x+1}$$

and so

$$\int \frac{x+2}{x^2+x}\,dx = 2\int \frac{dx}{x} - \int \frac{dx}{x+1}$$

$$= 2\ln|x| - \ln|x+1| + C$$

Ex4. $\displaystyle\int \frac{5x+4}{(x+2)^2(x-1)}\,dx$

Note Here we have a repeated factor $(x+2)^2$ in the denominator. However, we will need a $\frac{1}{x+2}$ term in our partial fractions expansion as well as a $\frac{1}{(x+2)^2}$ term.

So write

$$\frac{5x+4}{(x+2)^2(x-1)} = \frac{A}{x+2} + \frac{B}{(x+2)^2} + \frac{C}{x-1}$$

Common denominator

$$= \frac{A(x+2)(x-1)}{(x+2)^2(x-1)} + \frac{B(x-1)}{(x+2)^2(x-1)} + \frac{C(x+2)^2}{(x+2)^2(x-1)}$$

$$= \frac{A(x+2)(x-1) + B(x-1) + C(x+2)^2}{(x+2)^2(x-1)}.$$

$$= \frac{A(x^2+x-2) + B(x-1) + C(x^2+4x+4)}{(x+2)^2(x-1)}$$

$$= \frac{(A+C)x^2 + (A+B+4C)x - 2A - B + 4C}{(x+2)^2(x-1)}$$

We can now equate the numerators and compare like powers of x.

$x^2/ \quad A \quad + C = 0 \quad (1)$

$x/ \quad A + B + 4C = 5 \quad (2)$

$1/ \quad -2A - B + 4C = 4 \quad (3)$

Easiest thing here is to use 1 to deduce that $C = -A$ and subst. in 2, 3

$$A + B - 4A = 5$$
$$-2A - B - 4A = 4$$

Tidy

$$-3A + B = 5 \qquad 2'$$
$$-6A - B = 4 \qquad 2'$$

Add 2', 3' to get rid of B

$$-9A = 9$$

$$A = -1$$

Then, subsitituting in 2',

$$-3(-1) + B = 5$$

$$3 + B = 5$$

$$B = 2$$

Finally $C = -A = -(-1) = 1.$

Hence $A = -1,\ B = 2,\ C = 1$

and we have our partial fraction expansion

$$\frac{5x+4}{(x+2)^2(x-1)} = \frac{-1}{x+2} + \frac{2}{(x+2)^2} + \frac{1}{x-1}.$$

Thus

$$\int \frac{5x+4}{(x+2)^2(x-1)}\,dx = \int \left(\frac{-1}{x+2} + \frac{2}{(x+2)^2} + \frac{1}{x-1} \right) dx$$

$$= -\int \frac{dx}{x+2} + 2\int \frac{dx}{(x+2)^2} + \int \frac{1}{x-1}\,dx$$

$$= -\ln|x+2| - \frac{2}{(x+2)} + \ln|x-1| + C.$$

Ex 5

$$\int \frac{x^3 - 7x^2 + 10x + 1}{x^2 - 7x + 10}\, dx$$

Here the degree of the numerator is $\geqslant$ that of the denominator.

The first thing we have to do is algebraic long division

$$\begin{array}{r|l} & \;\; x \\ \hline x^2 - 7x + 10 & x^3 - 7x^2 + 10x + 1 \\ & x^3 - 7x^2 + 10x \\ \hline & \qquad\qquad\qquad 1 \end{array}$$

So

$$\frac{x^3 - 7x^2 + 10x + 1}{x^2 - 7x + 10} = \frac{x(x^2 - 7x^2 + 10) + 1}{x^2 - 7x + 10}$$

$$= x + \frac{1}{x^2 - 7x + 10}.$$

If we factor $x^2-7x+10$ as $(x-2)(x-5)$, we get

$$\frac{x^3-7x^2+10x+1}{x^2-7x+10} = x + \frac{1}{(x-2)(x-5)}$$

We then use partial fractions on $\frac{1}{(x-2)(x-5)}$.

From earlier

$$\frac{1}{(x-2)(x-5)} = -\frac{1}{3(x-2)} + \frac{1}{3(x-5)}$$

and so

$$\frac{x^3-7x^2+10x+1}{x^2-7x+10} = x - \frac{1}{3(x-2)} + \frac{1}{3(x-5)}$$

Thus

$$\int \frac{x^3 - 7x^2 + 10x + 1}{x^2 - 7x + 10}\, dx$$

$$= \int x\, dx - \frac{1}{3}\int \frac{dx}{x-2} + \frac{1}{3}\int \frac{dx}{x-5}$$

$$= \frac{x^2}{2} - \frac{1}{3}\ln|x-2| + \frac{1}{3}\ln|x-5| + C$$

Partial fractions works for many, but not all rational functions.

Ex 6 $\displaystyle\int \frac{2x^2+x+19}{(x^2+4)(x+1)}$

Here x^2+4 is a so-called irreducible quadratic factor which cannot be broken dan into the product of two linear factors. The correct partial fractions expansion to use here is

$$\frac{2x^2+x+19}{(x^2+4)(x+1)} = \frac{Ax+B}{x^2+4} + \frac{C}{(x+1)}$$

$$= \frac{(Ax+B)(x+1) + C(x^2+4)}{(x^2+4)(x+1)}$$

$$= \frac{(Ax^2+Ax+Bx+B) + (Cx^2+4C)}{(x^2+4)(x+1)}$$

$$= \frac{(A+C)x^2 + (A+B)x + B+4C}{(x^2+4)(x+1)}$$

Equate like powers of x.

x^2 / $\quad A + C = 2 \quad$ (1)

x / $\quad A + B = 1 \quad$ (2)

1 / $\quad B + 4C = 19 \quad$ (3)

(1) − (2) gives

$$-B + C = 1 \quad (4)$$

Add (3), (4)

$$\begin{aligned} B + 4C &= 19 \\ -B + C &= 1 \\ \hline 5C &= 20 \end{aligned}$$

So $C = 4$.

Subst for C in (1)

$$A + 4 = 2$$

$$A = -2$$

Subst for A in (2)

$$-2 + B = 1$$

$$B = 3$$

Hence $A = -2$, $B = 3$, $C = 4$ and

$$\frac{2x^2 + x + 19}{(x^2+4)(x+1)} = \frac{-2x+3}{x^2+4} + \frac{4}{x+1}$$

Integrating

$$\int \frac{2x^2+x+19}{(x^2+4)(x+1)}\,dx = \int \left(\frac{-2x+3}{x^2+4} + \frac{4}{x+1} \right) dx.$$

We need to split up the first piece into two separate integrals

$$= -\int \frac{2x}{x^2+4}\,dx + 3\int \frac{dx}{x^2+4} + 4\int \frac{dx}{x+1}$$

$$= -\ln|x^2+4| + \frac{3}{2}\arctan\left(\frac{x}{2}\right) + 4\ln|x+1| + C$$

(using an easy substitution $u = x^2+4$ on the first integral and a tangent substitution $x = 2\tan\theta$ on the second).

Ex.

$$\int \frac{10x - 2x^2}{(x-1)^2(x+3)}$$

NOTE Although the $(x-1)$ term is squared, we will need a $\frac{1}{x-1}$ term in our partial fraction expansion as well as a $\frac{1}{(x-1)^2}$ term

So write

$$\frac{10x - 2x^2}{(x-1)^2(x+3)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+3}.$$

$$= \frac{A(x-1)(x+3)}{(x-1)^2(x+3)} + \frac{B(x+3)}{(x-1)^2(x+3)} + \frac{C(x-1)^2}{(x-1)^2(x+3)}$$

$$= \frac{A(x^2+2x-3) + B(x+3) + C(x^2-2x+1)}{(x-1)^2(x+3)}$$

$$= \frac{(A+C)x^2 + (2A+B-2C)x + (-3A+3B+C)}{(x-1)^2(x+3)}$$

Compare powers of x.

$$\begin{array}{lllll} x^2/ & A & + C = & -2 & (1) \\ x/ & 2A + B & - 2C = & 10 & (2) \\ 1 & -3A + 3B & + C = & 0 & (3) \end{array}$$

Solve by Gaussian elimination

Take 2(1) from (2) and add 3(1) to (3) to get rid of A in (2) & (3)

$$\begin{array}{lll} A + C = -2 & (1) \\ B - 4C = 14 & (2)' \\ 3B + 4C = -6 & (3)' \end{array}$$

Take 3(2)' from (3)' to get rid of B in (3)'

$$\begin{array}{lll} A + C = -2 & (1) \\ B - 4C = 14 & (2)' \\ 16C = -48 & (3)'' \end{array}$$

Now do back substitution to get C, B, A in turn.

From (3)'', $C = -3$

sub for C into (2)'

$$B - 4(-3) = 14$$

$$B = 2$$

Sub for ~~B~~, C in (1)

$$A - 3 = -2$$

$$A = 1.$$

So $A = 1$, $B = 2$, $C = -3$ and

$$\frac{10x - 2x^2}{(x-1)^2(x+3)} = \frac{1}{x-1} + \frac{2}{(x-1)^2} - \frac{3}{x+3}$$

The actual integration is now fairly easy

$$\int \frac{10x - 2x^2}{(x-1)^2(x+3)}\,dx = \int \left(\frac{1}{x-1} + \frac{2}{(x-1)^2} - \frac{3}{x+3} \right) dx$$

$$= \int \frac{dx}{x-1} + 2\int \frac{dx}{(x-1)^2} - 3\int \frac{dx}{x+3}$$

In the first two integrals let

$w = x-1, \quad dw = dx$

and in the third let

$y = x+3, \quad dy = dx$

Get

$$\int \frac{dw}{w} + 2\int \frac{dw}{w^2} - 3\int \frac{dy}{y}$$

$$= \ln|w| + 2\,\frac{w^{-1}}{(-1)} - 3\ln|y| + C$$

Convert back to x and tidy

$$= \ln|x-1| - \frac{2}{(x-1)} - 3\ln|x+3| + C.$$

Ex.
$$\int \frac{2x^2 - x - 1}{(x^2+1)(x-2)}\,dx$$

Here x^2+1 cannot be factored any further (using only real numbers). The correct partial fractions expansion to look for in this case is of the form

$$\frac{2x^2 - x - 1}{(x^2+1)(x-2)} = \frac{Ax+B}{x^2+1} + \frac{C}{x-2}$$

$$= \frac{(Ax+B)(x-2) + C(x^2+1)}{(x^2+1)(x-2)}$$

$$= \frac{Ax^2 + (-2A+B)x - 2B + Cx^2 + C}{(x^2+1)(x-2)}$$

$$= \frac{(A+C)x^2 + (-2A+B)x - 2B + C}{(x^2+1)(x-2)}$$

Equating powers of x gives

x^2/

$$
\begin{aligned}
A \qquad\qquad\quad + C &= 2 \qquad (1) \\
-2A + B \qquad\qquad &= -1 \qquad (2) \\
-2B + C &= -1 \qquad (3)
\end{aligned}
$$

Gaussian elimination (again!)

Add 2(1) to (2) to get rid of A in (2), (3)

$$
\begin{aligned}
A \qquad\qquad + C &= 2 \qquad (1) \\
B + 2C &= 3 \qquad (2)' \\
-2B + C &= -1 \qquad (3)'
\end{aligned}
$$

Add 2(2)' to (3)' to get rid of B in (3)'

$$
\begin{aligned}
A \qquad\qquad + C &= 2 \qquad (1) \\
B + 2C &= 3 \qquad (2)' \\
5C &= 5 \qquad (3)''
\end{aligned}
$$

Back substitution

③'' $\Rightarrow$ $C = 1$

Sub for C in ②'

$$B + 2 = 3$$

$$B = 1$$

Sub for B, C in ①

$$A + 1 = 2$$

$$A = 1.$$

So $A = B = C = 1$ and

$$\frac{2x^2 - x - 1}{(x^2+1)(x-2)} = \frac{x+1}{x^2+1} + \frac{1}{x-2}.$$

Thus

$$\int \frac{2x^2 - x - 1}{(x^2+1)(x-2)}\,dx = \int \frac{x+1}{x^2+1}\,dx + \int \frac{dx}{x-2}$$

Split up the first integral further to get

$$= \int \frac{x\,dx}{x^2+1} + \int \frac{1 \cdot dx}{x^2+1} + \int \frac{dx}{x-2}$$

In the first integral let $w = x^2+1$,
so

$$dw = 2x\,dx \quad \text{and} \quad \frac{dw}{2} = x\,dx$$

Get

$$\frac{1}{2}\int \frac{dw}{w} + \int \frac{dx}{x^2+1} + \int \frac{dx}{x-2}$$

$$= \frac{1}{2}\ln|w| + \frac{1}{1}\arctan\left(\frac{x}{1}\right) + \ln|x-2| + C$$

Convert back to x

$$= \frac{1}{2}\ln|x^2+1| + \arctan x + \ln|x-2| + C.$$

Ex 7 $\displaystyle\int \frac{4x^3 + 2x^2 + x + 2}{x^4 + x^3}\, dx.$

In order to find the correct partial fractions expansion, we first need to factor the denominator

$$x^4 + x^3 = x^3(x+1)$$

The x^3 is a triple repeated factor and the necessary partial fractions expansion is thus

$$\frac{4x^3 + 2x^2 + x + 2}{x^4 + x^3} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x^3} + \frac{D}{x+1}$$

(again, those lower powers are necessary).

$$= \frac{Ax^2(x+1) + Bx(x+1) + C(x+1) + Dx^3}{x^3(x+1)}$$

$$= \frac{A(x^3 + x^2) + B(x^2 + x) + C(x+1) + Dx^3}{x^3(x+1)}$$

$$= \frac{(A + D)x^3 + (A + B)x^2 + (B + C)x + C}{x^3(x+1)}$$

Equating like powers of x:

x^3/	$A + D = 4$	①
x^2/	$A + B = 2$	②
x/	$B + C = 1$	③
1	$D = 2$	④

④ tells us $D = 2$ and substituting in ① gives

$A + 2 = 4$

$A = 2$

Substitute for A in (2)

$$2 + B = 2$$

$$B = 0$$

Subst for B in (3)

$$0 + C = 1$$

$$C = 1.$$

Hence $A = 2, \quad B = 0, \quad C = 1, \quad D = 2$

and

$$\frac{4x^3 + 2x^2 + x + 2}{x^4 + x^3} = \frac{2}{x} + \frac{1}{x^3} + \frac{2}{x+1}.$$

Integrating is now quite easy

$$\int \frac{4x^3 + 2x^2 + x + 2}{x^4 + x^3} \, dx = \int \left(\frac{2}{x} + \frac{1}{x^3} + \frac{2}{x+1} \right) dx$$

$$= 2\int \frac{dx}{x} + \int \frac{dx}{x^3} + 2\int \frac{dx}{x+1}$$

$$= 2\ln|x| - \frac{1}{x^2} + 2\ln|x+1| + C.$$

Strategy for Integrating a Rational Function $\frac{P(x)}{Q(x)}$.

1. If $\deg P \geq \deg Q$, use algebraic long division and use partial fractions on the remainder.

2. If $Q(x)$ is a product of distinct linear factors, use partial fractions of the form

$$\frac{A}{x-c}$$

3. If $Q(x)$ has a repeated linear factor $(x-c)^n$, use partial fractions of the form

$$\frac{A_1}{x-c} + \frac{A_2}{(x-c)^2} + \cdots + \frac{A_{n-1}}{(x-c)^n}$$

(the same n as the power of the factor itself)

4. If $Q(x)$ contains an irreducible quadratic factor $q(x)$, try a partial fraction of the form

$$\frac{Ax+B}{q(x)}$$

If this factor $q(x)$ is repeated twice so that $Q(x)$ contains $(q(x))^2$, you will need extra terms

$$\frac{Ax+B}{q(x)} + \frac{Cx+D}{(q(x))^2}$$

and so on.....