# §7.3 Integrating Trigonometric Functions

We examine how to do two types of integral

I. Integrals of powers of sine and cosine of the form

$$\int \sin^m x \cos^n x \, dx.$$

Here things are easier if at least one of $m, n$ is odd. The procedure is as follows.

$\underline{n \text{ odd}}$

- Split off a factor of $\cos x$ and put it with the $dx$
- Use $\cos^2 x = 1 - \sin^2 x$ to convert the remaining cosines to expressions involving sines.
- Make the substitution $u = \sin x$, $du = \cos x \, dx$

$\underline{m \text{ odd}}$

- Split off a factor of $\sin x$ and put it with the $dx$
- Use $\sin^2 x = 1 - \cos^2 x$ to convert the remaining sines into expressions involving cosines
- Make the substitution $u = \cos x$, $du = -\sin x \, dx$.

$m, n$ even · Use the half-angle formulae

$$\sin^2 x = \frac{1}{2}(1 - \cos 2x)$$

$$\cos^2 x = \frac{1}{2}(1 + \cos 2x)$$

to convert the integral into one involving lower powers of $\sin(2x)$ and $\cos(2x)$.

N.b. We may need to do the above reduction for $m, n$ both even more than once until we have enough odd powers.

**Ex 1**

$$\int \cos^3 x = \int \cos^2 x \cdot \cos x \, dx$$

$$= \int (1-\sin^2 x) \cos x \, dx$$

Let $u = \sin x$, $du = \cos x \, dx$

$$= \int (1-u^2) \, du$$

$$= u - \frac{u^3}{3} + C$$

$$= \sin x - \frac{\sin^3 x}{3} + C$$

**Ex 2**

$$\int \sin^2 x \, dx = \int \frac{1}{2} (1-\cos 2x) \, dx$$

$$= \frac{1}{2} \int (1-\cos 2x) \, dx$$

$$= \frac{x}{2} - \frac{\sin 2x}{4} + C$$

Similarly $\int \cos^2 x \, dx = \frac{x}{2} + \frac{\sin 2x}{4} + C$.

Ex 3 $\int \sin^5 x \cos^6 x \, dx$

$$= \int \sin^4 x \cos^6 x \cdot \sin x \, dx$$

$$= \int (\sin^2 x)^2 \cos^6 x \cdot \sin x \, dx$$

$$= \int (1-\cos^2 x)^2 \cos^6 x \cdot \sin x \, dx$$

Let $u = \cos x$, $du = -\sin x \, dx$, $-du = \sin x \, dx$

$$= \int (1-u^2)^2 u^6 \cdot -du$$

$$= -\int (1-2u^2+u^4) u^6 \cdot du$$

$$= -\int (u^6 - 2u^8 + u^{10}) \, du$$

– always multiply out before integrating!

$$= -\frac{u^7}{7} + \frac{2u^9}{9} - \frac{u^{11}}{11} + C$$

$$= -\frac{\cos^7 x}{7} + \frac{2\cos^9 x}{9} - \frac{\cos^{11} x}{11} + C.$$

Ex 4 $\int \sin^4 x \, dx$

Here $m, n$ both even, so we must first reduce

$$= \int (\sin^2 x)^2 \, dx$$

$$= \int \left(\frac{1}{2}(1-\cos 2x)\right)^2 dx$$

$$= \frac{1}{4} \int (1 - 2\cos 2x + \cos^2 2x) \, dx .$$

For the $\cos^2 2x$ part we must reduce again

$$= \frac{1}{4} \int \left(1 - 2\cos 2x + \frac{1}{2}(1 + \cos 4x)\right) dx$$

$(2 \times 2x = 4x)$

Tidy up before integrating

$$= \int \left( \frac{3}{8} - \frac{1}{2}\cos 2x + \frac{1}{8}\cos 4x \right) dx.$$

At last, we can integrate

$$= \frac{3x}{8} - \frac{1}{4}\sin 2x + \frac{1}{32}\sin 4x + C.$$

Ex 5 $\int \sin^4 x \cos^2 x \, dx$

Again $m, n$ both even, so we must reduce

$$= \int \left(\tfrac{1}{2}(1-\cos 2x)\right)^2 \cdot \tfrac{1}{2}(1+\cos 2x)\, dx$$

$$= \frac{1}{8}\int (1-2\cos 2x+\cos^2 2x)(1+\cos 2x)\, dx$$

Multiply out

$$= \frac{1}{8}\int (1-2\cos 2x+\cos^2 2x+\cos 2x$$
$$-2\cos^2 2x+\cos^3 2x)\, dx$$

$$= \frac{1}{8}\int (1-\cos 2x-\cos^2 2x+\cos^3 2x)\, dx$$

We need to reduce again for the $\cos^2 2x$ term

$$= \frac{1}{8}\int \left(1 - \cos 2x - \frac{1}{2}(1 + \cos 4x) + \cos^3 2x\right) dx$$

Tidy

$$= \frac{1}{8}\int \left(\frac{1}{2} - \cos 2x - \frac{1}{2}\cos 4x + \cos^3 2x\right) dx$$

The $\cos^3 2x$ part needs extra work (n odd case), so we,ll put that on its own

$$= \frac{1}{8}\left\{ \int \left(\frac{1}{2} - \cos 2x - \frac{1}{2}\cos 4x\right) dx \right.$$
$$\left. + \int \cos^3 2x \, dx \right\}$$

$$= \frac{1}{8}\left\{\int \left(\frac{1}{2} - \cos 2x - \frac{1}{2}\cos 4x\right) dx\right.$$

$$\left. + \int \cos^2 2x \cdot \cos 2x \; dx \right\}$$

$$= \frac{1}{8}\left\{\int \left(\frac{1}{2} - \cos 2x - \frac{1}{2}\cos 4x\right) dx\right.$$

$$\left. + \int (1 - \sin^2 2x) \cos 2x \; dx \right\}$$

Let $u = \sin 2x$, $du = 2\cos 2x$

$$\frac{du}{2} = \cos 2x$$

$$= \frac{1}{8}\left\{\int \left(\frac{1}{2} - \cos 2x - \frac{1}{2}\cos 4x\right) dx\right.$$

$$\left. + \frac{1}{2}\int (1 - u^2)\, du \right\}$$

Now we can integrate!

$$= \frac{1}{8}\left\{ \frac{x}{2} - \frac{\sin 2x}{2} - \frac{\sin 4x}{8} \right.$$

$$\left. + \frac{u}{2} - \frac{u^3}{6} \right\} + C$$

Convert back to $x$

$$= \frac{1}{8}\left\{ \frac{x}{2} - \cancel{\frac{\sin 2x}{2}} - \frac{\sin 4x}{8} + \cancel{\frac{\sin 2x}{2}} - \frac{\sin^3 2x}{6} \right\}$$

$$+ C$$

$$= \frac{x}{16} - \frac{\sin 4x}{64} - \frac{\sin^3 2x}{6} + C$$

And that's as bad as it gets for sine/cosine integrals!

II Integrals of powers of tangent and secant of the form

$$\int \tan^m x \sec^n x \, dx$$

Before outlining the general procedure, we deal with two elementary cases

__Ex.6__ $\int \tan x = \int \frac{\sin x}{\cos x} \, dx$

$u = \cos x, \quad du = -\sin x \, dx$

$$= \int -\frac{du}{u}$$

$$= -\int \frac{du}{u}$$

$$= -\ln|u| + C$$

$$= -\ln|\cos x| + C$$

$$= \ln|\sec x| + C$$

as $\ln\left(\frac{1}{y}\right) = -\ln y$ and $\sec = \frac{1}{\cos}$

Ex7 $\displaystyle\int \sec x \, dx$

$$= \int \sec x \left( \frac{\sec x + \tan x}{\sec x + \tan x} \right) dx$$ Trick!

$$= \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x} \, dx$$

$u = \sec x + \tan x$

$du = (\sec x \tan x + \sec^2 x) \, dx$

$\quad = (\sec^2 x + \sec x \tan x) \, dx$

Substitute for $u$, $du$

$$= \int \frac{du}{u}$$

$= \ln|u| + C$

$= \ln|\sec x + \tan x| + C.$

General procedure for $\int \tan^m x \sec^n x \, dx$ (when $m+n > 1$).

<u>n even</u>

- Split off a factor of $\sec^2 x$ and put it with the $dx$
- Use $\sec^2 x = 1 + \tan^2 x$ to convert the remaining secants into expressions involving $\tan x$.
- Make the substitution $u = \tan x$, $du = \sec^2 x \, dx$

<u>m odd</u>

- Split off a factor of $\sec x \tan x$ and put it with the $dx$.
- Use $\tan^2 x = \sec^2 x - 1$ to convert the remaining tangents into expressions involving $\sec x$.
- Make the substitution $u = \sec x$, $du = \sec x \tan x \, dx$.

m even, n odd — The hard case.

Use $\tan^2 x = \sec^2 x - 1$ to convert the integral to an expession in secants only. You will then need integration by parts to handle integrating higher powers of $\sec x$.

N.b. the book suggests using a reduction formula, which is more or less the same thing in the end.

## Ex 8

$$\int \tan^2 x \sec^4 x \, dx$$

Here $n = 4$ is even, so we rewrite as

$$= \int \tan^2 x \sec^2 x \cdot \sec^2 x \, dx$$

$$= \int \tan^2 x \,(1 + \tan^2 x) \cdot \sec^2 x \, dx$$

Now let $u = \tan x$, so $du = \sec^2 x \, dx$

$$= \int u^2 (1 + u^2) \, du$$

$$= \int (u^2 + u^4) \, du$$

$$= \frac{u^3}{3} + \frac{u^5}{5} + C$$

$$= \frac{\tan^3 x}{3} + \frac{\tan^5 x}{5} + C$$

Ex 9 $\quad \int \tan^3 x \sec^4 x \, dx$

Here $m$ is odd and $n$ even, so we have a choice.

To be different, we'll do it using $m$ odd

$$= \int \tan^2 x \sec^3 x \cdot \sec x \tan x \, dx$$

$$= \int (\sec^2 x - 1) \sec^3 x \sec x \tan x \, dx$$

Now let $u = \sec x$, $du = \sec x \tan x \, dx$

$$= \int (u^2 - 1) u^3 \, du$$

$$= \int (u^5 - u^3) \, du$$

$$= \frac{u^6}{6} - \frac{u^4}{4} + C$$

$$= \frac{\sec^6 x}{6} - \frac{\sec^4 x}{4} + C$$

Ex 10 $\int \tan^2 x \sec x \, dx$

Here $m$ is even and $n$ odd, so we first convert to $\sec x$ and then use integration by parts to integrate higher powers of $\sec x$.

$$= \int (\sec^2 x - 1) \sec x \, dx$$

$$= \int (\sec^3 x - \sec x) \, dx$$

$$= \int \sec^3 x \, dx - \int \sec x \, dx.$$

We integrate $\sec^3 x$ by parts

$$\int \sec^3 x \, dx = \int \sec x \sec^2 x \, dx$$

$u = \sec x \qquad , \quad dv = \sec^2 x \, dx$

$du = \sec x \tan x \, dx, \quad v = \tan x.$

So

$$\int \sec^3 x \, dx = \sec x \tan x - \int \tan x \sec x \tan x \, dx$$

$$= \sec x \tan x - \int \tan^2 x \sec x \, dx$$

$$= \sec x \tan x - \int (\sec^2 x - 1) \sec x \, dx$$

$$= \sec x \tan x - \int \sec^3 x \, dx + \int \sec x \, dx$$

Adding $\int \sec^3 x \, dx$ to both sides so it disappears on the rhs.

$$2 \int \sec^3 x \, dx = \sec x \tan x + \int \sec x \, dx$$

and then

$$\int \sec^3 x \, dx = \frac{\sec x \tan x}{2} + \frac{1}{2} \int \sec x \, dx$$

$$= \frac{\sec x \tan x}{2} + \frac{1}{2} \ln |\sec x + \tan x| + C.$$

Substituting back for $\int \sec^3 x \, dx$

$$\int \tan^2 x \sec x \, dx = \int \sec^3 x \, dx - \int \sec x \, dx$$

$$= \frac{\sec x \tan x}{2} + \frac{1}{2} \ln |\sec x + \tan x|$$

$$- \ln|\sec x + \tan x| + C$$

$$= \frac{\sec x \tan x}{2} - \frac{\ln|\sec x + \tan x|}{2} + C$$