§ 6.6 Work

Many problems in physics are solved by calculating integrals.

Work/Energy

If a constant force F is applied to move an object a distance d, we define the work W done to be the product

$$W = F \cdot d.$$

Units

In metric (SI) units, the unit of force is the Newton, N $\quad 1\,N = 1\,kg\,m/s^2$.

The unit of work/energy is the Joule J

$$1\,J = 1\,Nm = 1\,kg\,m^2/s^2.$$

In imperial (English) units, the unit of force is the pound, lb and the unit of work is the foot pound ft lb.

Ex1 If it takes 50 lbs of force to move a box 10 ft across the floor of my basement, the work I do is

$$50 \times 10 = 500 \text{ ft lbs.}$$

Weight

Weight is the force exerted on an object due to the Earth's gravity acting on its mass.

In metric units, the weight of an object of mass m kg is

$$mg \quad \text{Newtons}$$

where $g = 9.81\,m/s^2$ is the acceleration due to gravity at the Earth's surface.

In English units, the pound is a unit of <u>force</u> (not of mass), so when we say an object is 10 lbs, we mean that the weight is 10 lbs, ie. the Earth's gravitational field exerts 10 lbs of force on it (n.b. one doesn't often talk of mass in English units - can you think why this might be so?).

A good example of the difference between mass and weight is to imagine standing on the surface of the moon whose gravity is six times weaker than Earth's. You <u>weight</u> is $\frac{1}{6}$ that of your weight on Earth, although your <u>mass</u> is unchanged.

<u>Ex2</u> I lift a 2 kg book from the floor onto a shelf 2m above the floor. The amount of work I do is

$$2\,kg \times 9.8\,m/s^2 \times 2\,m = 38.4\,J.$$

Work against a variable force.

If the force varies with the distance x, the expression

$$W = F \cdot d$$

is replaced with the integral

$$W = \int_a^b F(x)\,dx .$$

Measures the work done using a variable force $F(x)$ to move the object (in a straight line) from $x = a$ to $x = b$.

Ex3 Hooke's Law.

If a spring has natural length l and we extend it to a length l', then the force exerted by the spring (i.e. the tension) is given by

$$F = k(l' - l)$$

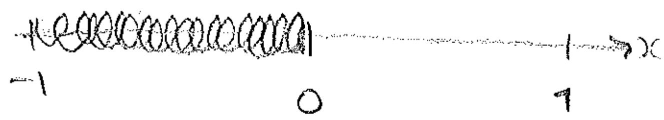
where k is a constant of proportionality called the <u>spring constant</u>.

The bigger k is, the stiffer the spring.

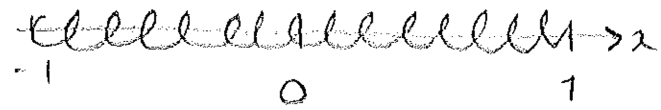
e.g. Suppose we have a spring of natural length 1 m and spring constant 4 which we then extend to 2 m.

Place the spring on the x-axis so that in the unextended position, the right end of the spring sits at $x = 0$.

Unextended



Extended



The force exerted when the right end of the spring sits at x $(0 \leq x)$ is then

$$k(l' - l)$$

$$= k(1 + x - 1)$$

$$= kx$$

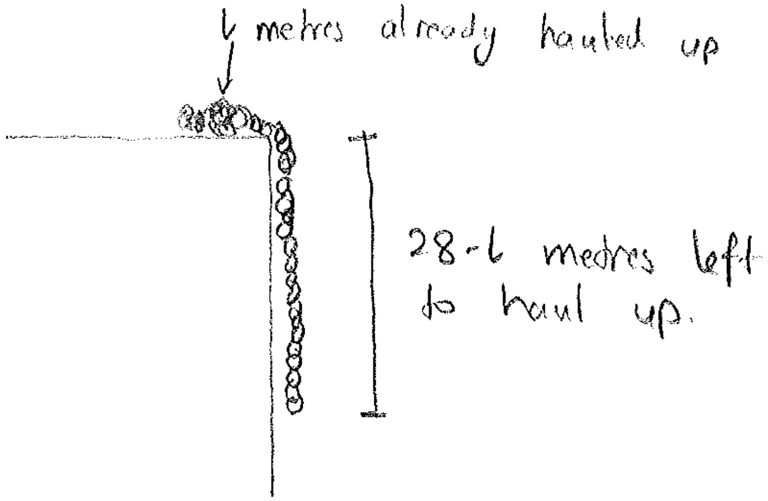
$$= 4x \quad \text{as} \quad k = 4.$$

The work done is then

$$W = \int_a^b F(x)\,dx = \int_0^1 4x\,dx = \left[2x^2\right]_0^1 = 2\,\text{J}.$$

Ex 4 A 28m uniform chain with a linear density λ of 2kg/m is dangling from the roof of a building. How much work is needed to pull the chain up onto the top of the building.

Let l represent the amount of chain already hauled up.



At this pt $28-l$ metres are left to haul up and this amount of chain has mass

$$2\,\text{kg/m} \times (28-l)\,\text{m} = 56-2l \text{ kg}$$

and weight

$$(56-2l)\cdot 9.8 \text{ Newtons.}$$

The work done to haul this piece of chain up a short distance Δl is then approx the weight $\times \Delta l$, ie.

$$\Delta W \approx (56 - 2l) \cdot 9.8 \, \Delta l \quad J.$$

Now add all these contributions ΔW to get

$$W \approx \sum (56 - 2l)(9.8) \, \Delta l$$

As $\Delta l \to 0$, this Riemann sum becomes an integral and we have

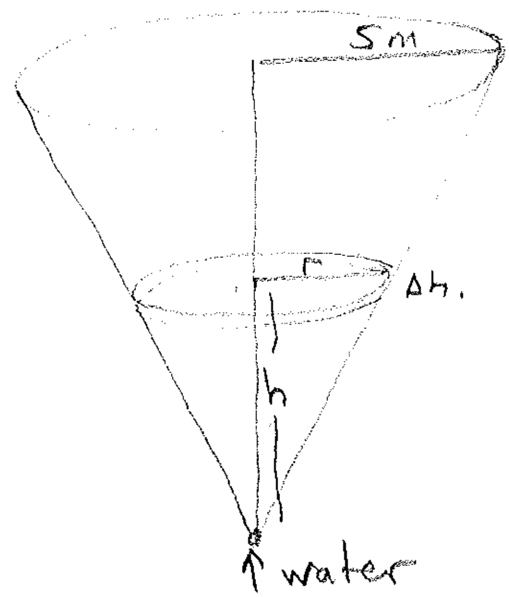
$$W = \int_0^{28} (56 - 2l)(9.8) \, dl$$

$$= 9.8 \left[56l - l^2 \right]_0^{28}$$

$$= 7{,}683.2 \quad J.$$

N.b. The book solves this differently by first splitting up the chain and then calculating the work needed to raise each small piece to the top.

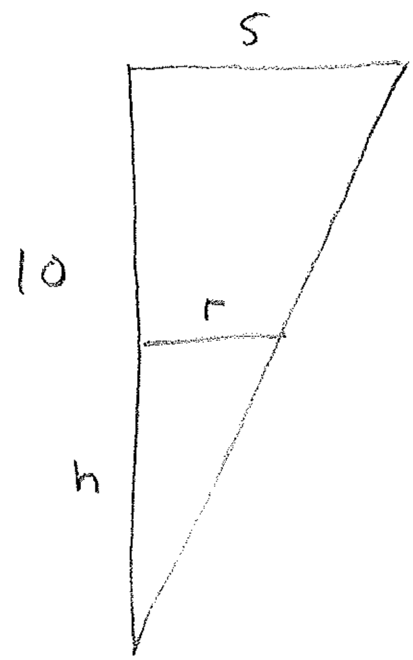
Ex. 5 A conical tank as shown is filled with water from its base. If the density of water is $1000 \, kg/m^3$, find the amount of work required to completely fill the tank.



Let h be the depth of water in the tank. The top of the water is approx. a cylinder of radius r and thickness Δh. and the volume of this piece of water is then approx

$$\pi r^2 \Delta h.$$

By similar triangles



$$\frac{r}{h} = \frac{5}{10} = \frac{1}{2}$$

$$r = h/2$$

Thus the volume of this slice of water is approx

$$\pi\left(\frac{h}{2}\right)^2 \Delta h = \frac{\pi h^2}{4}\Delta h.$$

The mass is then approx.

$$1000\,\frac{\pi h^2}{4}\,\Delta h = 250\pi h^2 \Delta h.$$

while the weight is

$$(9.81)(250\pi h^2)\Delta h$$

To fill the tank, this slice of water needs to be raised from the base of the tank to a height h and so the work done is approx

$$\Delta W = (9.81)(250\pi h^2)\Delta h \circ h$$

$$= (9.81)(250\pi h^3)\Delta h.$$

Summing up we get

$$W \approx \sum 9.81(250\pi h^3)\Delta h$$

and in the limit as $\Delta h \to 0$

$$W = \int_0^{10} (9.81)(250\pi h^3)\, dh$$

$$= (9.81)(250\pi)\int_0^{10} h^3\, dh$$

$$= (9.81)(250\pi)\left[\frac{h^4}{4}\right]_0^{10}$$

$$= (9.81)(250\pi)\left(\frac{10{,}000}{4}\right)$$

$$\approx 1.926 \times 10^{7} \text{ J}$$

or 19.26 MJ.

A person using a steady 250W of power to operate a hand pump would take about 21.4 hours to fill this tank.

Ex 6 The Space Elevator

A space elevator connects a pt. on the equator with geostationary orbit which is approx 36,000 km above that pt.

Calculate the amount of work the elevator has to do in order to lift a 70 kg person from the equator to geostationary orbit. If the elevator is powered by electricity which costs 10¢ / kWhr, how much does this cost?