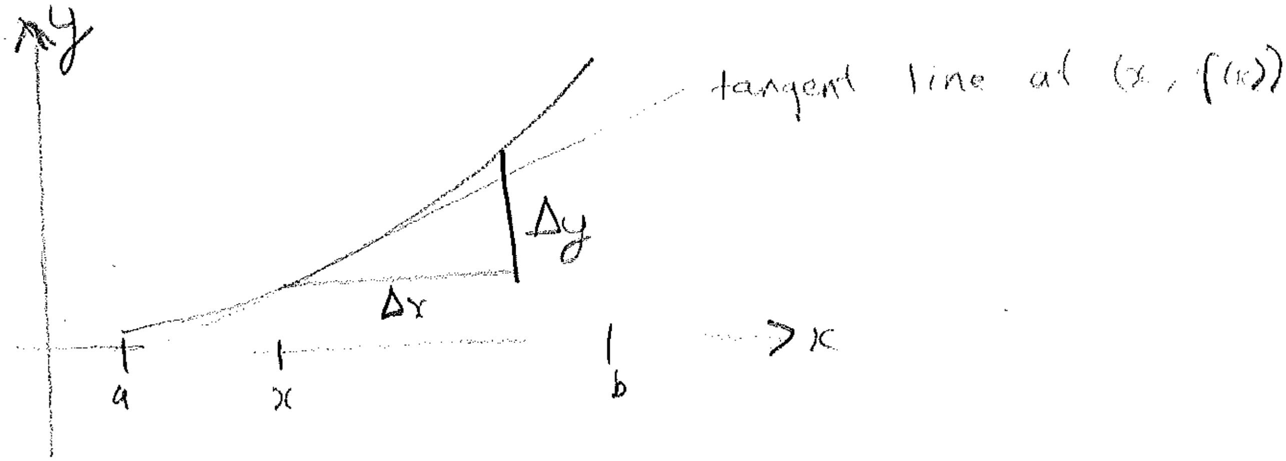
§6.4 Arc Length

Suppose we want to find the length, s, of the curve $y = f(x)$, $a \leq x \leq b$



If we approximate a small segment of the curve by a line segment, we get that the length of this part of the curve is, using Pythagoras, approx

$$\Delta s \approx \sqrt{(\Delta x)^2 + (\Delta y)^2}$$

Using the tangent line approximation (§ 3.9),

$$\Delta y \approx f'(x)\Delta x.$$

Thus

$$\Delta s \approx \sqrt{(\Delta x)^2 + (f'(x))^2 (\Delta x)^2}$$

$$= \sqrt{(1 + (f'(x))^2)(\Delta x)^2}$$

$$= \sqrt{(1 + (f'(x))^2} \, \Delta x.$$

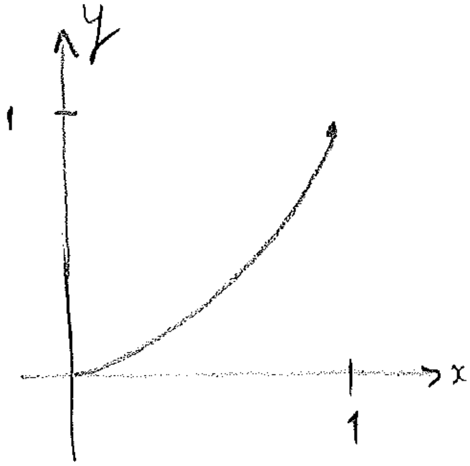
This leads to a Riemann sum

$$s \approx \sum_{i=1}^{n} \sqrt{(1 + (f'(x_{i-1}))^2} \, \Delta x.$$

Taking limits as $n \to \infty$, we define the arc length s by

$$s = \int_a^b \sqrt{1 + (f'(x))^2} \, dx.$$

Ex $\quad f(x) = \frac{2}{3}x^{3/2}, \quad 0 \le x \le 1$



$$f'(x) = x^{\frac{1}{2}}$$

$$\sqrt{1+(f'(x))^2} = \sqrt{1+(\sqrt{x})^2} = \sqrt{1+x}.$$

Thus

$$S = \int_0^1 \sqrt{1+(f'(x))^2}\,dx$$

$$= \int_0^1 \sqrt{1+x}\,dx$$

[If you like, let
$w = 1+x, \quad dw = dx$
when $x=0, w=1, x=1, w=2$]

$$= \left[\frac{2}{3}(1+x)^{3/2}\right]_0^1$$

$$= \frac{2}{3}(2^{3/2}-1).$$

Note: It is quite hard to find examples for arc-length where $\sqrt{1+(f'(x))^2}$ has an elementary antiderivative.