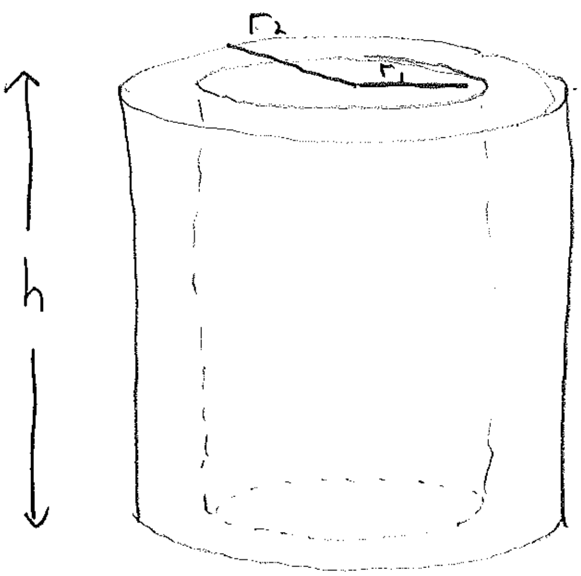
§ 6.3 Volumes by Cylindrical Shells

A cylindrical shell is a solid enclosed between two right circular cylinders



If the inner cylinder has radius r_1, the outer cylinder has radius r_2 while both have height h, then the volume V of the shell is given by

$V =$ volume of outer cylinder
  $-$ volume of inner cylinder

$$= \pi r_2^2 h - \pi r_1^2 h$$

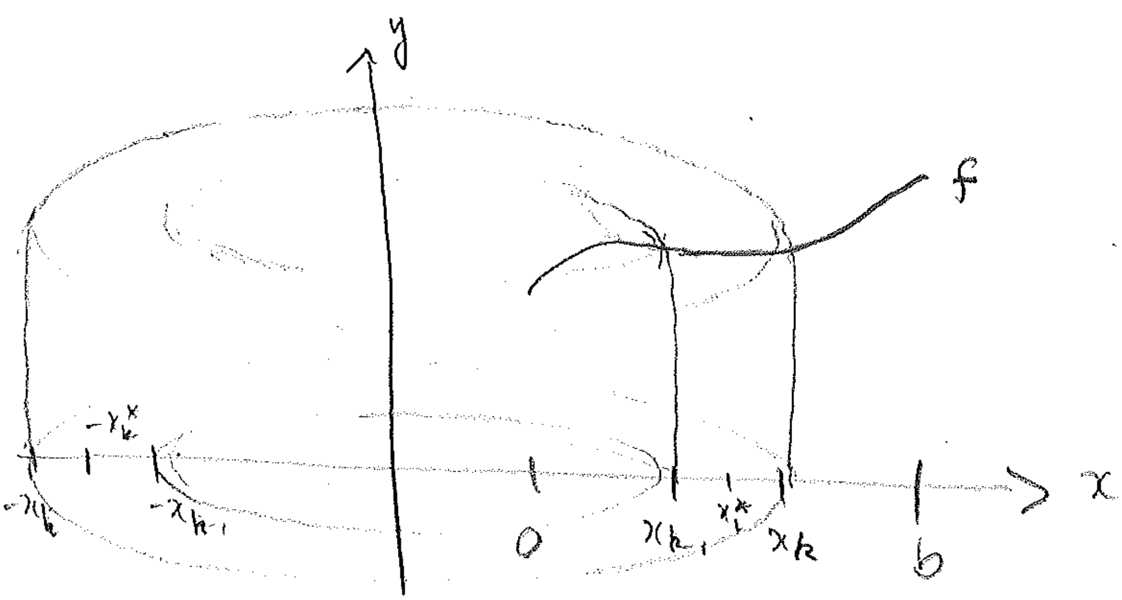
$$= \pi (r_2^2 - r_1^2) h$$

$$= \pi (r_2 + r_1)(r_2 - r_1) h$$ by difference of two squares

$$= 2\pi \cdot \frac{1}{2}(r_1 + r_2) \cdot h \cdot (r_1 - r_2)$$

$$= 2\pi \text{ (average radius)} \cdot \text{height} \cdot \text{thickness}.$$

Now suppose f is cts on the interval $[a, b]$, where $0 \leq a \leq b$ and we rotate the region under the graph of f about the y-axis.



For each interval $[x_{k-1}, x_k]$, the portion of the volume of revolution which corresponds to this interval is approximately a cylindrical shell of inner radius x_{k-1}, outer radius x_k and height $f(x_k^*)$.

The thickness of such a shell is

$$x_k - x_{k-1} =: \Delta x_k$$

while the average radius is $\frac{x_{k-1} + x_k}{2}$.

If we then let x_k^* be this midpoint $\frac{x_{k-1} + x_k}{2}$, then the volume of the shell is

$$2\pi \cdot x_k^* \cdot f(x_k^*) \cdot \Delta x_k .$$

Adding up these contributions for each interval gives an approximate total volume

$$V \approx \sum_{k=1}^{n} 2\pi x_k^* f(x_k^*) \Delta x_k .$$

Taking the limit as $n \to \infty$, these Riemann sums converge to the integral

$$V = \int_a^b 2\pi x f(x)\, dx .$$

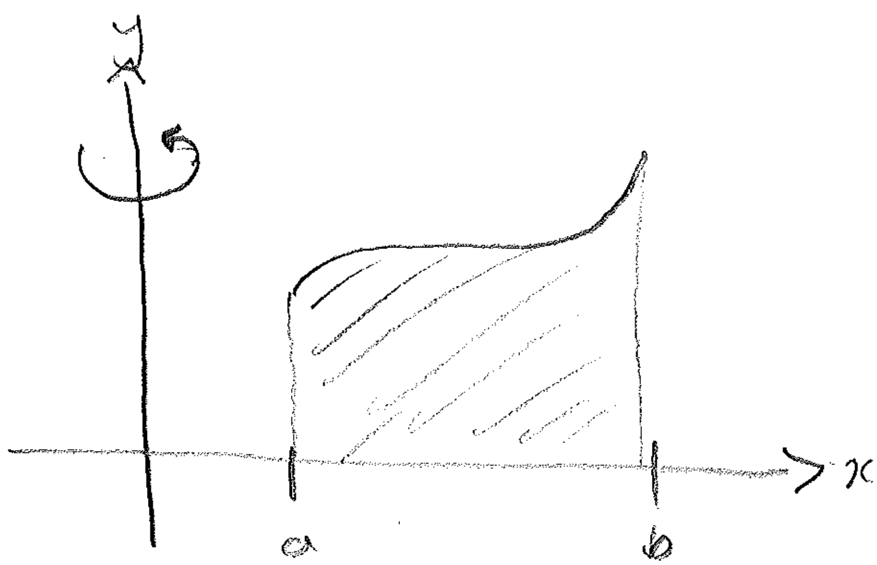
where we use this limit to <u>define</u> the volume of revolution.

In summary

Let f be cts. and ≥ 0 on the interval $[a,b]$ where $0 \leq a < b$.

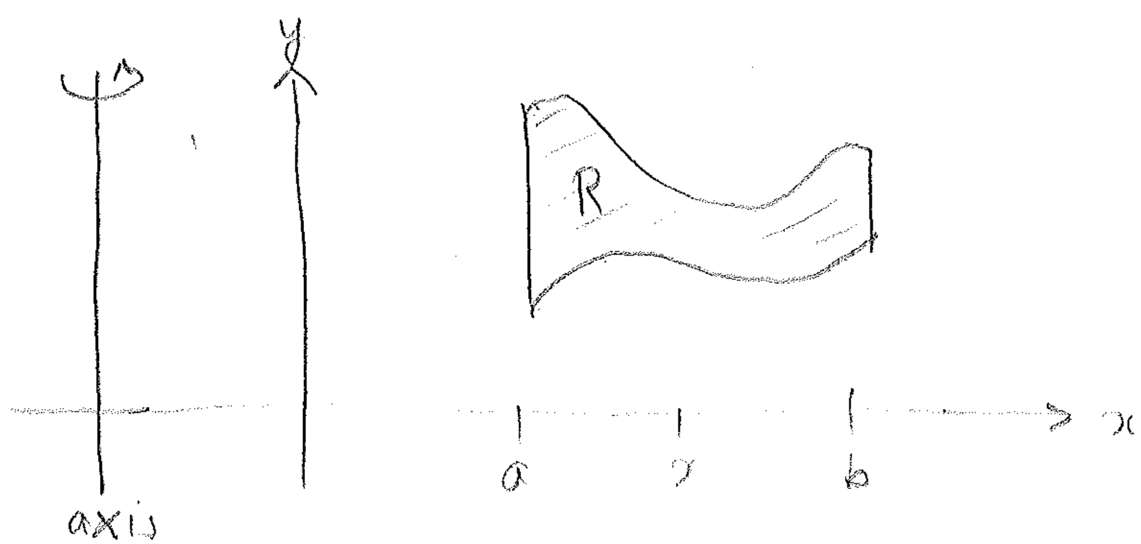
Then the volume of revolution obtained by rotating the region under the graph of f for $a \leq x \leq b$ about the y-axis is given by

$$V = \int_a^b 2\pi x f(x)\,dx$$



More generally, if we revolve some region R between $x = a$ and $x = b$ about some (vertical) axis, we have

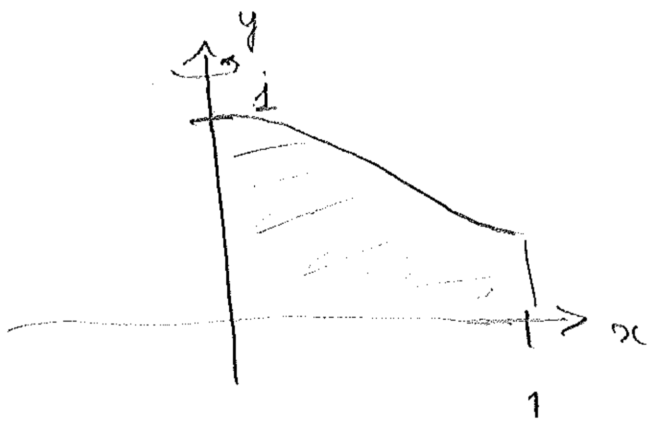
$$V = \int_a^b 2\pi \times \text{radius} \times \text{height}\, dx$$



where the radius as a fn of x is the distance from x to the axis of rotation.

Of course, similar formulae exist for horizontal axes of rotation where we have integrals in y rather than x.

Ex 1 Find the volume obtained by revolving the region under the graph of $f(x) = \frac{1}{1+x^2}$, $0 \leq x \leq 1$ about the y-axis.



Here

$$V = \int_0^1 2\pi x \, f(x)\, dx$$

$$= \int_0^1 \frac{2\pi x}{1+x^2}\, dx$$

$$= \pi \int_0^1 \frac{2x}{1+x^2}\, dx = \pi \left[\ln|1+x^2| \right]_0^1$$

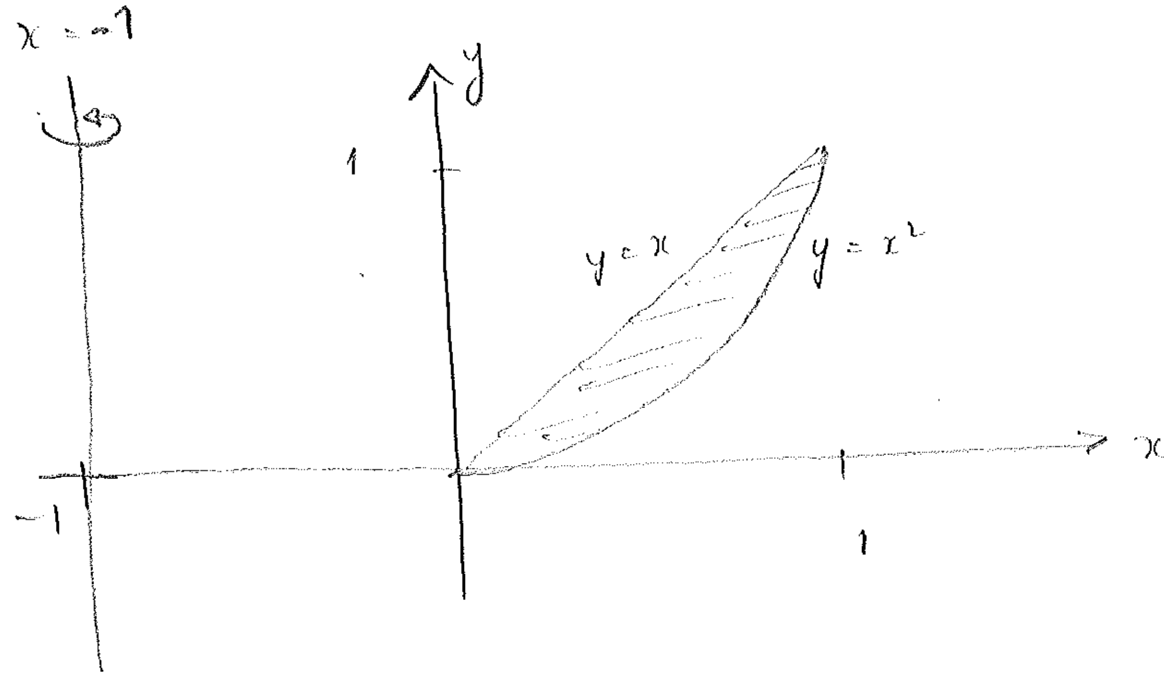
(easy substitution $u = 1+x^2$)

$$= \pi (\ln 2 - \ln 1)$$

$$= \pi \ln 2.$$

Ex 2 (A little different from the book on p. 435).

Find the volume obtained by revolving the region enclosed between the graphs of $y = x$ and $y = x^2$ about the line $x = -1$.



The first thing we need to do is determine our limits of integration.
As before with the area between two curves, we set the two curves equal to each other and solve for x.

$$x^2 = x$$

$$x^2 - x = 0$$

$$x(x-1) = 0$$

$$x = 0, 1$$

So $a = 0$, $b = 1$.

Clearly, for $0 \leq x \leq 1$, $x^2 \leq x$ and the height of our shell is then

$$f(x) = x - x^2.$$

The radius of the shell is $x+1$ (distance x to the y-axis and a further distance 1 to get to $x = -1$).

The volume of the shell is then

$$V = \int_0^1 2\pi \text{ radius} \times \text{height } dx$$

$$= \int_0^1 2\pi (1+x)(x - x^2)\, dx$$

$$= 2\pi \int_0^1 (x - x^2 + x^2 - x^3)\, dx$$

$$= 2\pi \int_0^1 (x - x^3)\, dx$$

$$= 2\pi \left[\frac{x^2}{2} - \frac{x^4}{4} \right]_0^1$$

$$= 2\pi \left(\left(\frac{1}{2} - \frac{1}{4} \right) - 0 \right)$$

$$= \frac{\pi}{2}.$$