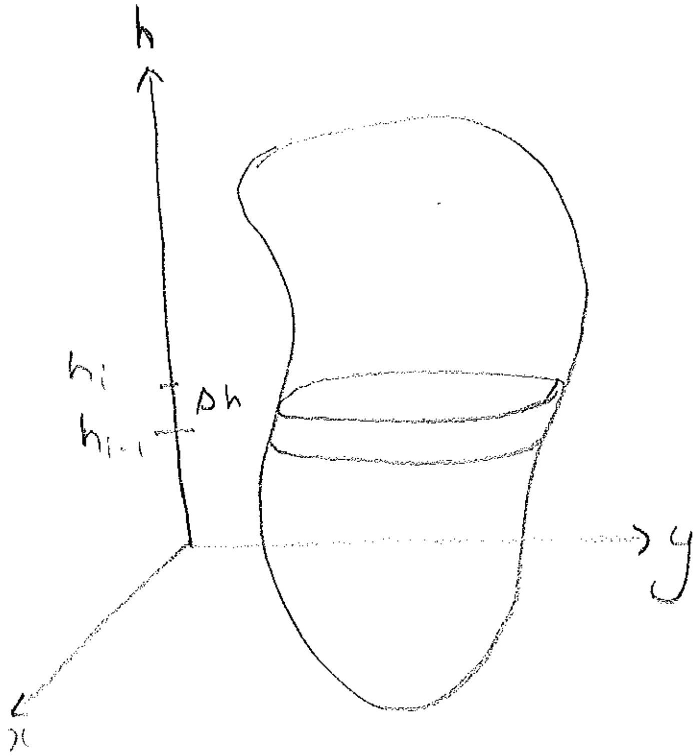
§ 6.2 Volumes by Slicing

Consider the figure below where the area $A = A(h)$ of the horizontal cross-section depends on the height h



Small horizontal piece is approx. a slice of volume

$$A(h_{i-1})\,\Delta h$$

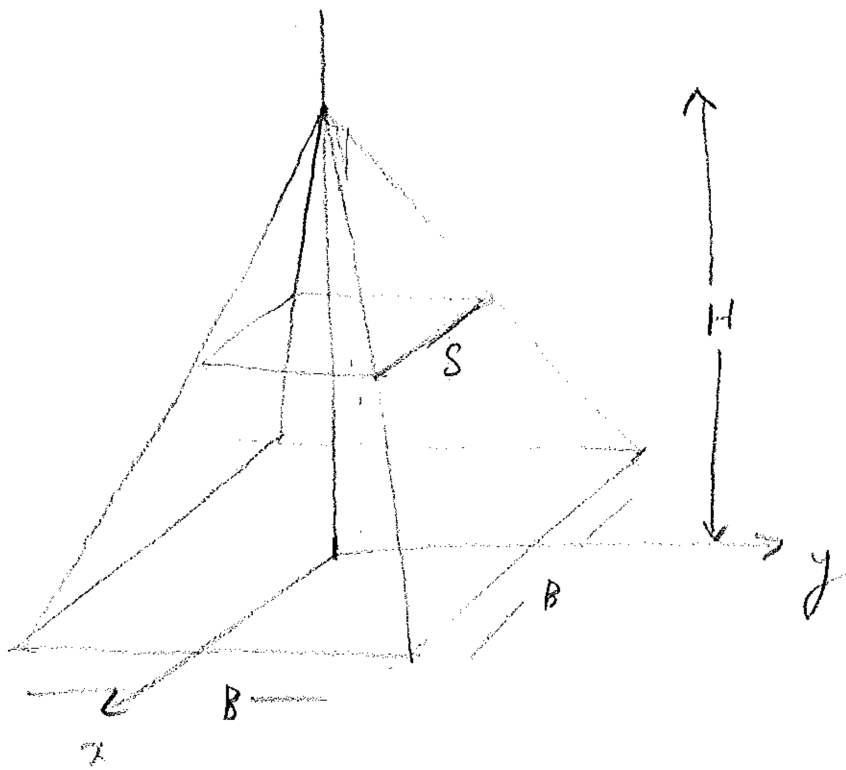
Add the slices together to get an approx. for the total volume V

$$V \approx \sum_{i=1}^{n} A(h_{i-1})\,\Delta h.$$

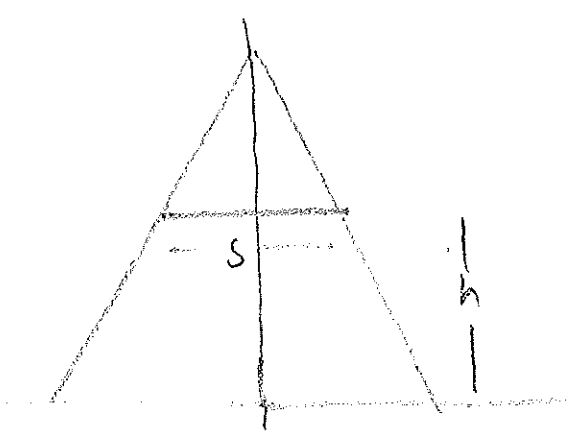
In the limit this Riemann sum becomes an integral and we define V by

$$V = \int_a^b A(h)\,dh$$

Ex. Pyramid with square base of side length B and height H.



Let $s = s(h)$ be the side length of the square cross-section at height h.



$s(h)$ is a linear fn of h where

$s(0) = B$ and $s(h) = 0$.

Implies that $$s(h) = B - \frac{B}{H} \cdot h$$

$$= B(1 - h/H)$$

Then $$A(h) = s^2 = B^2 (1 - h/H)^2$$

$$= B^2 \left(1 - \frac{2h}{H} + h^2/H^2\right)$$

and

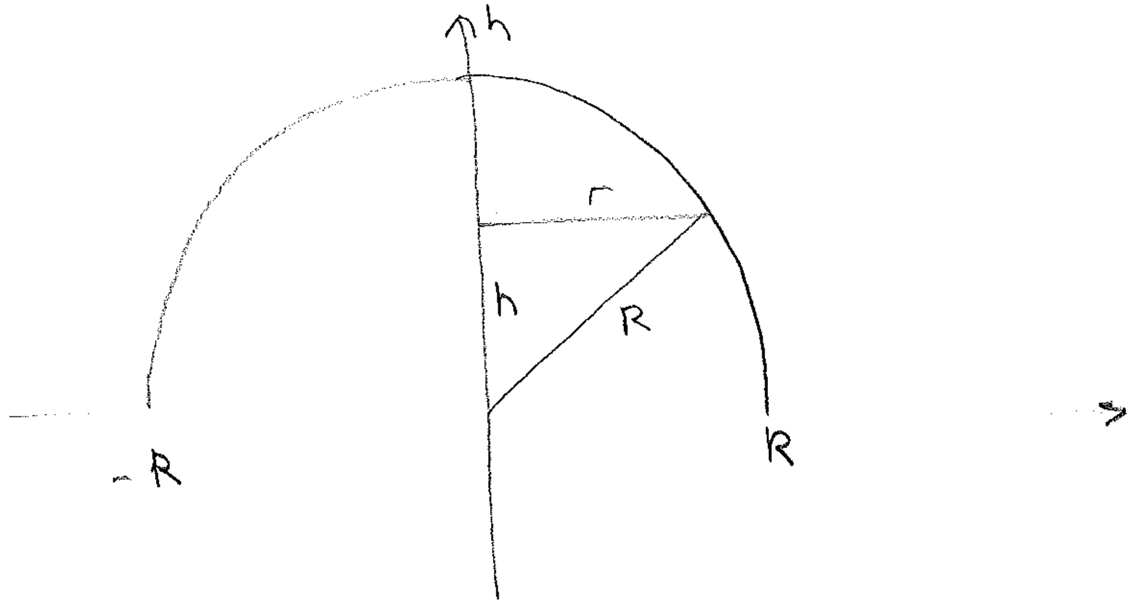
$$V = \int_0^H B^2 \left(1 - \frac{2h}{H} + \frac{h^2}{H^2}\right) dh$$

$$= B^2 \int_0^H \left(1 - \frac{2h}{H} + \frac{h^2}{H^2}\right) dh$$

$$= B^2 \left[h - \frac{h^2}{H} + \frac{h^3}{3H^2} \right]_0^H$$

$$= B^2 \left(\left(H - \frac{H^2}{H} + \frac{H^3}{3H^2} \right) - 0 \right) = \frac{B^2 H}{3}.$$

Ex. Hemisphere of radius R



Similarly do before

$$r = \sqrt{R^2 - h^2}$$

$$A(h) = \pi r^2 = \pi(R^2 - h^2)$$

$$V = \int_0^R A(h)\,dh = \pi \int_0^R (R^2 - h^2)\,dh$$

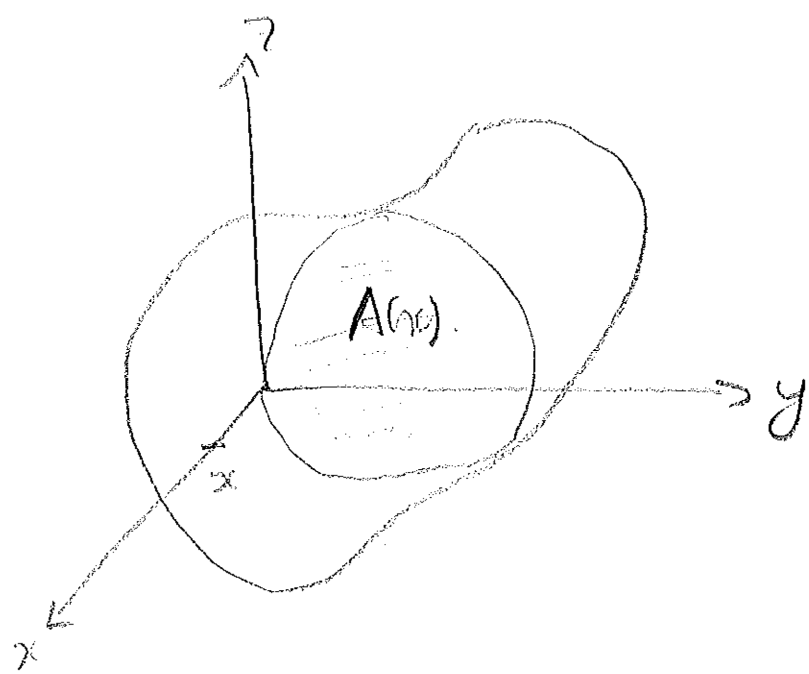
$$= \pi \left[R^2 h - \frac{h^3}{3} \right]_0^R$$

$$= \pi\left(\left(R^3 - R^3/3\right) - 0\right)$$

$$= \frac{2\pi R^3}{3}$$ as we'd expect.

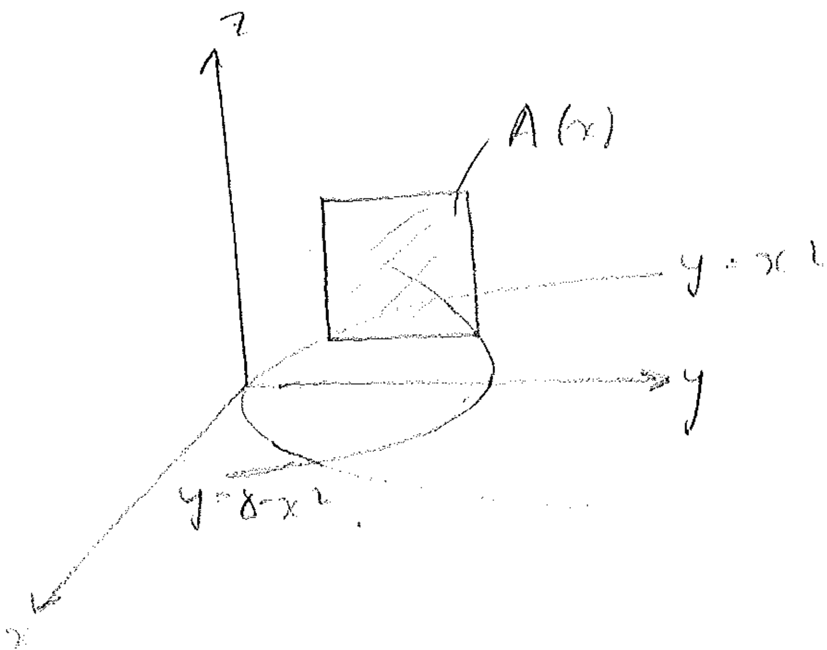
Volume by Vertical Cross-Sections

If we slice the region by vertical slices perpendicular to the x-axis whose area is $A(x)$, we get a similar formula



$$V = \int_a^b A(x)\,dx.$$

Ex. Find the volume of the solid whose base is the region in the x-y plane bounded by the parabolas $y = x^2$ and $y = 8 - x^2$ and whose cross-sections are squares perpendicular to the x-axis with one side in the x-y plane



As in previous example, x runs from -2 to 2. Also, the square has side length

$$s = 8 - 2x^2.$$

Then $A(x) = s^2 = (8-2x^2)^2$

$$= 64 - 32x^2 + 4x^4.$$

$$V = \int_{-2}^{2} (64 - 32x^2 + 4x^4)\,dx$$

$$= \left[64x - \frac{32}{3}x^3 + \frac{4}{5}x^5 \right]_{-2}^{2}$$

$$= 64(2) - \frac{32(8)}{3} + \frac{4}{5}(32)$$

$$- \left(64(-2) - \frac{32(-8)}{3} + \frac{4}{5}(-32) \right)$$

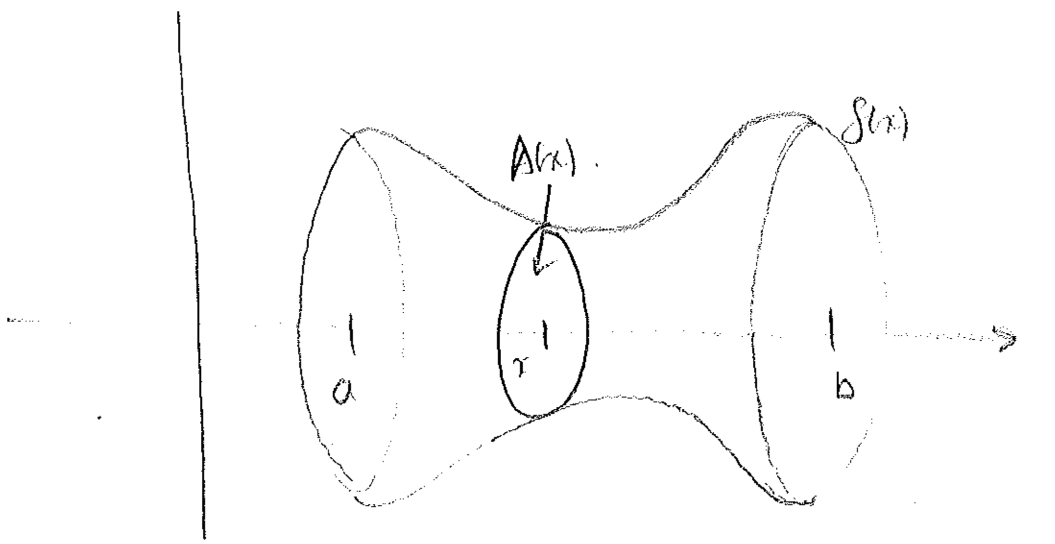
$$= 128 - \frac{256}{3} + \frac{128}{5}$$

$$+ 128 - \frac{256}{3} + \frac{128}{5}$$

$$= \frac{2048}{15} \approx 136.5.$$

Volumes of Revolution

Suppose we take a fn $f(x)$ which is $\geqslant 0$ on $[a, b]$ and rotate it about the axis. The curve sweeps out a solid with round cross-sections (like a lathe). Called a <u>volume</u> of revolution

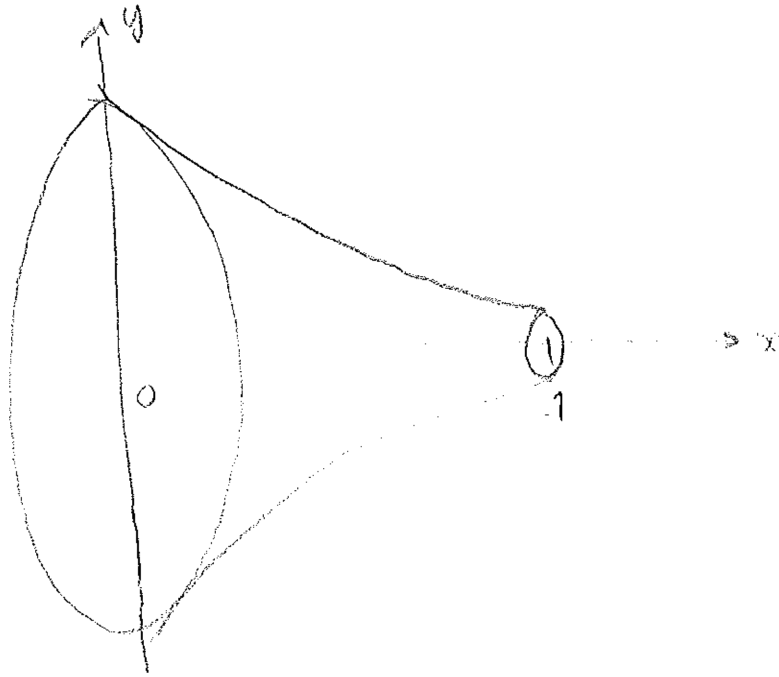


Here $A(x) = \pi (f(x))^2$, so

$$V = \int_a^b A(x)\,dx = \pi \int_a^b (f(x))^2\,dx = \pi \int_a^b (r(x))^2\,dx$$

Similar formulae exist for rotating about other lines (e.g. y-axis, $x = 3$).

Ex. The curve $y = e^{-x}$, $0 \leq x \leq 1$ is rotated about the x-axis.



Here $a = 0$, $b = 1$, $f(x) = e^{-x}$ and

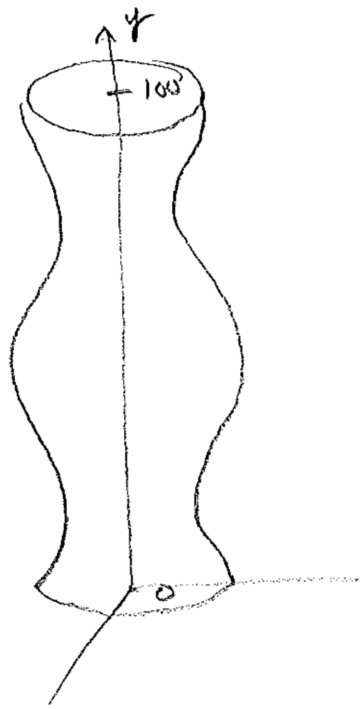
$$V = \pi \int_0^1 (f(x))^2 \, dx$$

$$= \pi \int_0^1 (e^{-x})^2 \, dx$$

$$= \pi \int_0^1 e^{-2x} \, dx = \pi \left[-\frac{1}{2} e^{-2x} \right]_0^1 = \pi \left(-\frac{1}{2} e^{-2} - \left(-\frac{1}{2} \cdot 1 \right) \right)$$

$$= \frac{\pi}{2} \left(1 - e^{-2} \right) \approx 1.36$$

Ex. Table leg is obtained by rotating $r = 3 + \cos(\pi y/25)$ cm $0 \le y \le 100$ cm about the y-axis



$$V = \pi \int_0^{100} (r(y))^2 \, dy$$

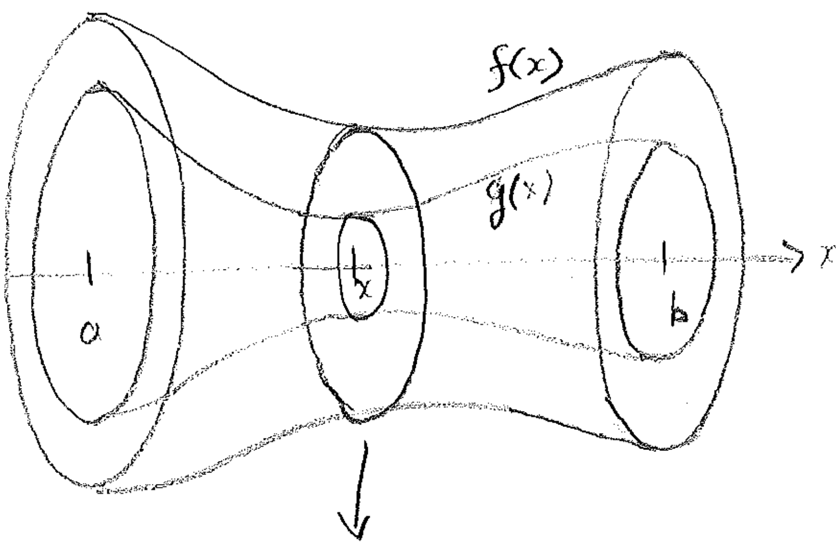
$$= \pi \int_0^{100} (3 + \cos(\pi y/25))^2 \, dy$$

$$= \pi \int_0^{100} (9 + 6\cos(\pi y/25) + \cos^2(\pi y/25)) \, dy$$

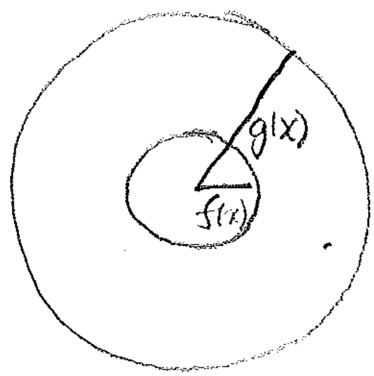
$$\approx 2984.5 \text{ cm}^3.$$

Check the (boring) details of the integration as an exercise!

If $0 \le g(x) \le f(x)$ on $[a,b]$ and we rotate the region between these curves about the x-axis, we have a volume whose cross-sections are annuli (rings).



Annulus.



Here

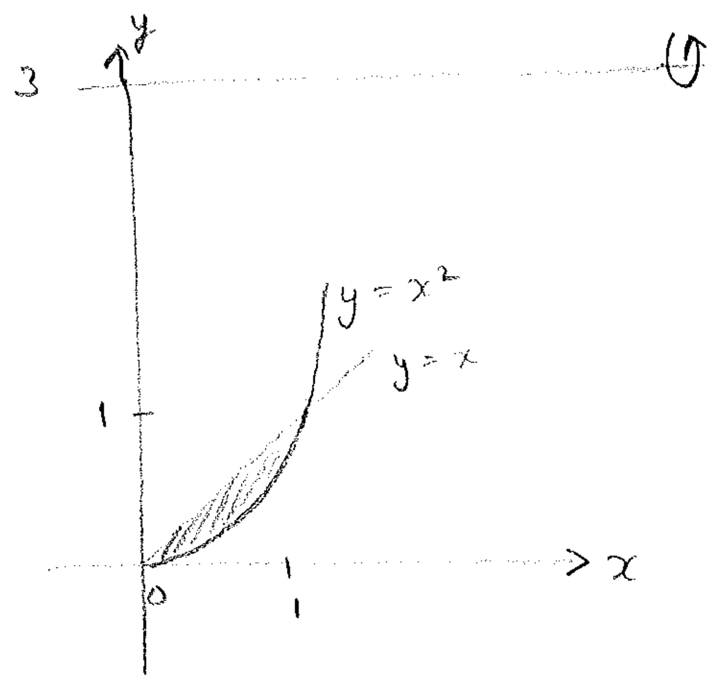
$$A(x) = \pi (f(x))^2 - \pi (g(x))^2$$
$$= \pi \left((f(x))^2 - (g(x))^2 \right)$$

Then

$$V = \pi \int_a^b \left((f(x))^2 - (g(x))^2 \right) dx$$

Again, similar formulae exist for rotating about other lines.

Ex 3. Region enclosed between $y = x$ and $y = x^2$ is rotated about the horizontal line $y = 3$.



Need to intersect the two curves to determine the limits of integration a, b.

So set

$$x = x^2$$

$$x - x^2 = 0$$

$$x(1 - x) = 0$$

$$x = 0, 1$$

So $a = 0$, $b = 1$.

Also $y = x^2$ is clearly the curve which is furthest from the axis of rotation

(e.g. $3 - (\frac{1}{2})^2 > 3 - \frac{1}{2}$), so here

$f(x) = 3 - x$

$g(x) = 3 - x^2$.

$$V = \pi \int_0^1 ((3 - x^2)^2 - (3 - x))^2 \, dx$$

$$= \pi \int_0^1 (9 - 6x^2 + x^4 - (9 - 6x + x^2)) \, dx$$

$$= \pi \int_0^1 (6x - 7x^2 + x^4) \, dx$$

$$= \pi \left[3x^2 - \frac{7x^3}{3} + \frac{x^5}{5} \right]_0^1$$

$$= \pi \left(3 - \frac{7}{3} + \frac{1}{5} - 0 \right)$$

$$= \pi \left(\frac{45 - 35 + 1}{15} \right) = \frac{11\pi}{15}$$