Chapter 6

Applications of the Definite Integral

§ 6.1 The Area between Two Curves

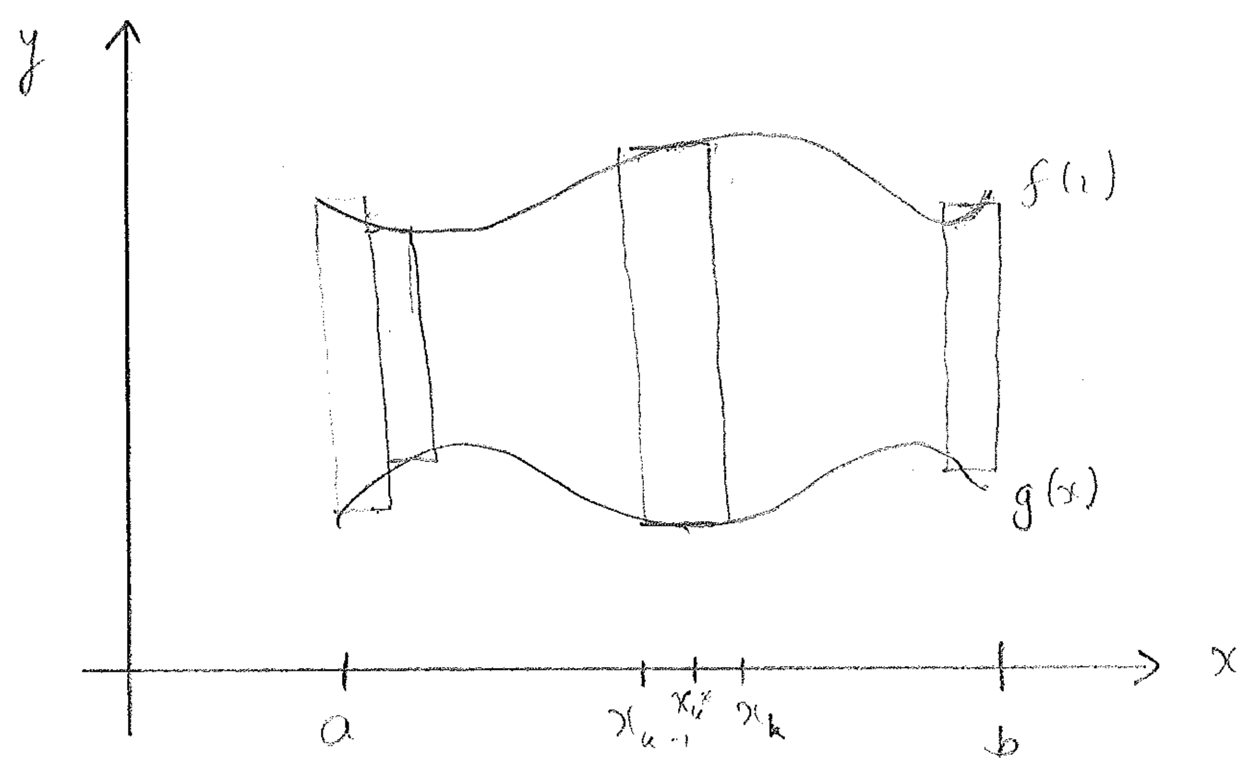
Suppose we have two functions f, g which are continuous (cts.) on an interval $[a, b]$ and that $f(x) \geq g(x) \quad \forall x \in [a, b]$

Now partition the interval $[a, b]$, i.e. take points x_i, $0 \leq i \leq n$ with

$$a = x_0 < x_1 < \quad \cdot \quad < x_{n-1} < x_n = b.$$

For each such interval $[x_{k-1}, x_k]$, $1 \leq k \leq n$, let $\Delta x_k = x_k - x_{k-1}$ be the width of this interval and let x_k^* be a sample point chosen from this interval.

The region between the graphs of f and g can then be approximated by a collection of skinny rectangles, one over each interval.



Each rectangle has width

$$\Delta x_k = x_k - x_{k-1}$$

and height $f(x_k^*) - g(x_k^*)$ given in terms of the values of f and g at the sample point x_k^*.

The area of this rectangle is then

$$(f(x_k^*) - g(x_k^*))\,\Delta x_k.$$

Summing up the areas of all these rectangles gives us an approximation to the area between the graphs of f and g.

$$A \approx \sum_{k=1}^{n} (f(x_k^*) - g(x_k^*))\,\Delta x_k.$$

We recognize that this is a Riemann sum for the function $f-g$ over $[a,b]$.

Since f, g are cts, so is $f-g$ and this fn is then integrable on $[a,b]$ and so the limit of these Riemann sums as $\max \Delta x_k \to 0$ exists.

By taking this limit as $\max \Delta x_k \to 0$, we then <u>define</u> the area between the graphs of f and g to be

$$A = \lim_{\max \Delta x_k \to 0} \sum_{k=1}^{n} \left(f(x_k^*) - g(x_k^*) \right) \Delta x_k$$

$$= \int_a^b \left(f(x) - g(x) \right) dx .$$

In summary.

<u>Area Formula</u> If f, g are ct's. on $[a, b]$ and $f(x) \geq g(x)$ $\forall x \in [a, b]$, then the area of the region between the graphs of f and g and between the lines $x = a$ and $x = b$ is

$$A = \int_a^b (f(x) - g(x))\, dx.$$

The main difficulty with using this formula is usually determining the correct limits a, b as the next example illustrates.

Ex 1. Find the area of the region between the graphs of $f(x) = x+6$ and $g(x) = x^2$ for $0 \leq x \leq 2$.

Note that for $0 \leq x \leq 2$,

$$x^2 \leq 2^2 = 4 < 6 \leq x+6$$

So $g(x) \leq f(x)$ on $[0, 2]$.

Then

$$A = \int_0^2 (f(x) - g(x))\,dx$$

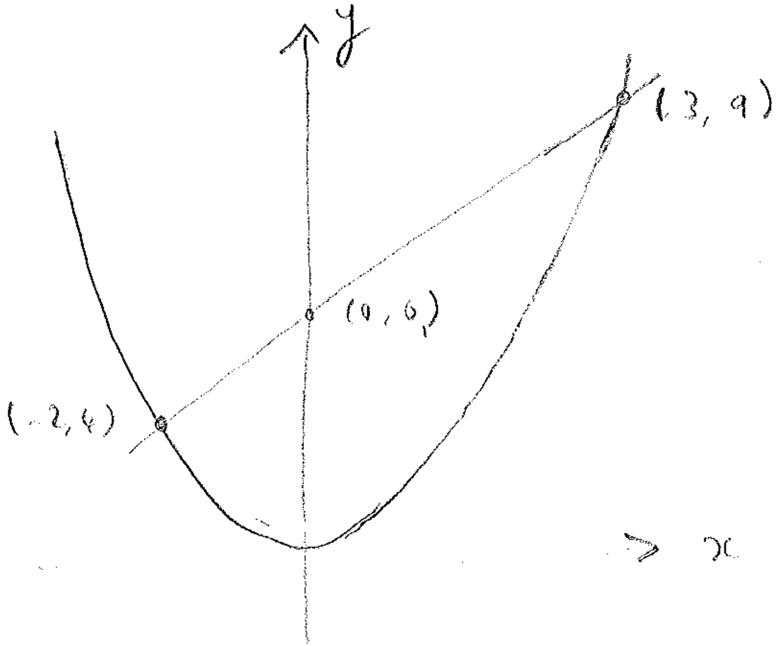
$$= \int_0^2 ((x+6) - x^2)\,dx$$

$$= \left[\frac{x^2}{2} + 6x - \frac{x^3}{3}\right]_0^2$$

$$= \frac{4}{2} + 12 - \frac{8}{3} - 0 = 14 - \frac{8}{3}$$

$$= \frac{42}{3} - \frac{8}{3} = \frac{34}{3}.$$

Ex 2. Find the area of the region enclosed between the curves $y = x^2$ and $y = x + 6$



In order to find the limits of integration, we need to know where the two curves meet.

To do this we set them equal to each other and solve for x.

$$x^2 = x + 6$$

$$x^2 - x - 6 = 0$$

Factor

$$(x - 3)(x + 2) = 0$$

So $x = 3, -2$ and our integral runs from $a = -2$ to $b = 3$.

On $[-2, 3]$, the line is the bigger fn, e.g. at $x = 0$, $x + 6 = 6$
$x^2 = 0$.

Hence

$$A = \int_{-2}^{3} \left((x+6) - x^2 \right) dx$$

$$= \int_{-2}^{3} (x + 6 - x^2)\, dx.$$

$$= \left[\frac{x^2}{2} + 6x - \frac{x^3}{3} \right]_{-2}^{3}$$

$$= \frac{9}{2} + 18 - \frac{27}{3}$$

$$- \left(\frac{4}{2} - 12 - \frac{(-8)}{3} \right))$$

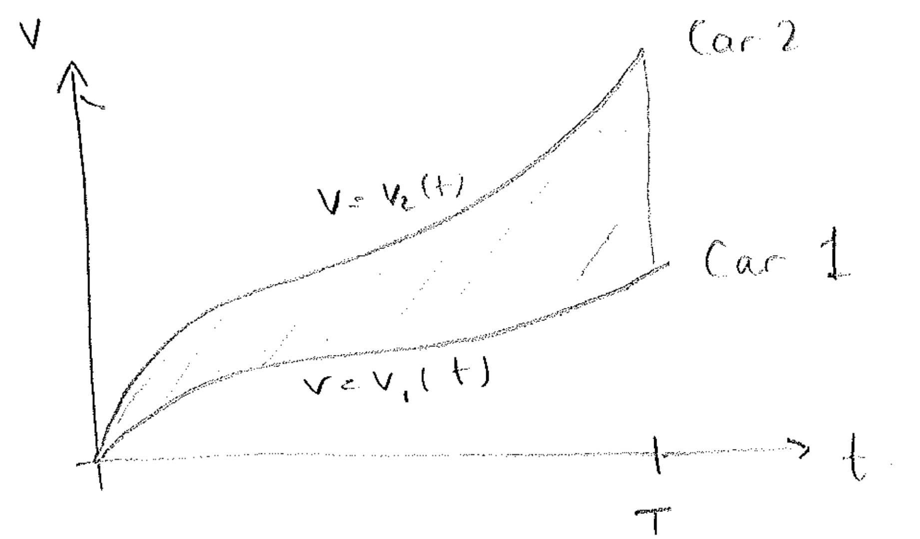
$$= \frac{9}{2} + 18 - 9 - \left(-10 + \frac{8}{3} \right)$$

$$= \frac{9}{2} + 9 + 10 - \frac{8}{3}$$

$$= \frac{27}{6} + \frac{54}{6} + \frac{60}{6} - \frac{16}{6}$$

$$= \frac{125}{6}.$$

Ex 3 Two race cars start out from the same position at $t = 0$ and their velocity versus time graphs are as below.



Give a physical interpretation of the area between the two graphs over the interval $[0, T]$.

On $[0, T]$, $v_1(t) \geq 0$, so

$$\int_0^T v_1(t)\,dt$$

represents the distance travelled by Car 1.

Similarly $\int_0^T v_2(t)\,dt$ represents the distance travelled by car 2.

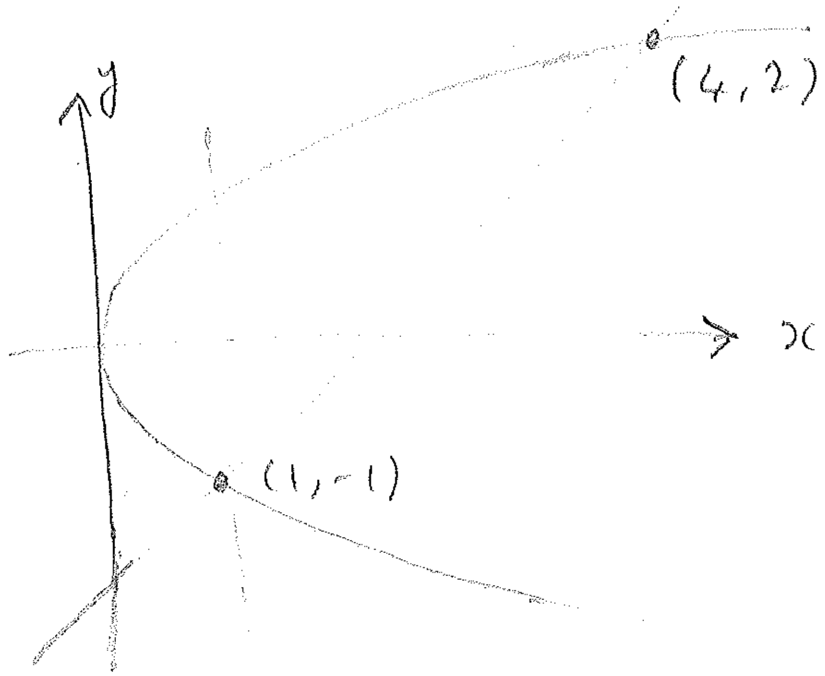
Since $v_1(t) \le v_2(t)$ for $0 \le t \le T$, Car 2 is always ahead of Car 1, while the area between the graphs

$$A = \int_0^T (v_2(t) - v_1(t))\,dt$$

$$= \int_0^T v_2(t)\,dt - \int_0^T v_1(t)\,dt$$

represents the distance Car 2 is ahead of Car 1 at time $t = T$.

Ex 4.5 Find the area of the region enclosed between $x = y^2$ and $y = x - 2$.



We first do this as an integral in x.

From the picture the region consists of two pieces - one between the two branches of the parabola and one between the line and the parabola.

Our first task is to find where the two curves meet.

We have $x = y^2$

$$y = x + 2 \Rightarrow y^2 = (x-2)^2$$
$$= x^2 - 4x + 4.$$

Hence, equating for y^2

$$x^2 - 4x + 4 = x$$

$$x^2 - 5x + 4 = 0$$

Factoring

$$(x-1)(x-4) = 0$$

So $x = 1, 4$.

For the part of the region between the two branches of the hyperbola, $0 \leq x \leq 1$ and the region lies between

$g(x) = -\sqrt{x}$ and $f(x) = +\sqrt{x}$.

This gives an area

$$\int_0^1 (\sqrt{x} - (-\sqrt{x}))\,dx = \int_0^1 2\sqrt{x}\,dx$$

$$= 2\int_0^1 x^{\frac{1}{2}}\,dx$$

$$= 2\left[\frac{2}{3}x^{3/2}\right]_0^1$$

$$= 2\left(\frac{2}{3} - 0\right) = \frac{4}{3}.$$

For the part of the region between the line and the parabola, x runs from 1 to 4. The parabola $y = +\sqrt{x}$ is the larger fn, e.g. at $x = 2$.

$$g(x) = x - 2 = 0$$
$$f(x) = +\sqrt{x} = \sqrt{2}.$$

This gives an area

$$\int_1^4 (\sqrt{x} - (x-2))\,dx$$

$$= \int_1^4 (x^{\frac{1}{2}} - x + 2)\,dx$$

$$= \left[\frac{2}{3}x^{3/2} - \frac{x^2}{2} + 2x\right]_1^4$$

$$= \frac{2}{3}(4)^{3/2} - \frac{16}{2} + 8 - \left(\frac{2}{3} - \frac{1}{2} + 2\right)$$

$$= \frac{2}{3}(8) - 8 + 8 - \left(\frac{1}{6} + 2\right)$$

$$= \frac{16}{3} - 2 - \frac{1}{6} = \frac{10}{3} - \frac{1}{6} = \frac{19}{6}.$$

Combining these two pieces gives a total area of

$$\frac{4}{3} + \frac{19}{6} = \frac{27}{6} = \frac{9}{2}.$$

However, this problem is much easier if we do this an integral in y rather than x.

Here $x = y^2$, $y = x - 2 \Rightarrow x = y + 2$.

Equating to find the points of intersection

$$y^2 = y + 2$$

$$y^2 - y - 2 = 0$$

$$(y - 2)(y + 1) = 0$$

So $y = 2, -1$ and our area can be found using a single integral from -1 to 2.

On $[-1, 2]$, $x = y + 2$ is the larger fn, e.g.

$$0^2 = 0$$

$$0 + 2 = 2.$$

Hence the area is given by

$$A = \int_{-1}^{2} ((y+2) - y^2)\, dy$$

$$= \left[\frac{y^2}{2} + 2y - \frac{y^3}{3} \right]_{-1}^{2}$$

$$= \frac{4}{2} + 4 - \frac{8}{3} - \left(\frac{1}{2} + 2(-1) - \frac{(-1)}{3} \right)$$

$$= 2 + 4 - \frac{8}{3} - \frac{1}{2} + 2 - \frac{1}{3}$$

$$= 8 - \frac{1}{2} - \frac{9}{3}$$

$$= 5 - \frac{1}{2} = \frac{9}{2} \quad \text{as before.}$$

In general, if $v(y)$, $w(y)$ are cts fns in y on the interval $[c,d]$ and $v(y) \leq w(y)$ $\forall y \in [c,d]$, then the area of the region between the graphs of v and w and between the lines $y=c$ and $y=d$ is

$$A = \int_c^d (w(y) - v(y))\,dy$$