# §5.9 Evaluating Definite Integrals by Substitution

Suppose we have a fn $g(x)$ whose derivative $g'(x)$ is continuous (cts.) on a closed interval $[a,b]$ and that we have another fn $f$ which is cts. on an interval $I$ containing the values of $g(x)$ for $a \leq x \leq b$. (i.e. $g([a,b]) \subset I$).

This setup allows us to say that the fn. $f(g(x)) \cdot g'(x)$ is cts on $[a,b]$.

Also, since $f$ is cts on $I$,
$f$ has an antid. $F$ on $I$.

If we then let $u = g(x)$ and apply the chain rule,

$$\frac{d}{dx}(F(g(x))) = F'(g(x)) \cdot g'(x)$$
$$= f(g(x)) \cdot g'(x), \quad a \leq x \leq b.$$

i.e. $F(g(x))$ is an antid of $f(g(x)) \cdot g'(x)$ on $[a,b]$.

Then, by the first part of the Fundamental Theorem of Calculus (F T o C), we get.

$$\int_a^b f(g(x)) \cdot g'(x)\,dx = \Big[F(g(x))\Big]_a^b$$

$$= F(g(b)) - F(g(a))$$

$$= \int_{g(a)}^{g(b)} f(u)\,du.$$

We have proved.

5-9.1 Theorem If $g'$ is cts on $[a, b]$ and $f$ is cts on an interval $I$ containing the values of $g(x)$, $a \le x \le b$, then

$$\int_a^b f(g(x))\, g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du.$$

This result gives us two basic ways to evaluate definite integrals using substitution.

The first method uses.

$$\int_a^b f(g(x)) \cdot g'(x)\,dx = \left[ \int f(g(x)) \cdot g'(x)\,dx \right]_a^b$$

ie use ordinary substitution (without limits) to first find an antiderivative

— easier conceptually but usually more work.

The second method involves making the substitution $u = g(x)$ in the definite integral and then

This result gives us two basic ways to evaluate definite integrals using substitution.

1. Use substitution to find the indefinite integral

$$\int f(g(x)) \cdot g'(x)\, dx$$

and then evaluate the definite integral using

$$\int_a^b f(g(x)) \cdot g'(x)\,dx = \left[ \int f(g(x)) \cdot g'(x)\,dx \right]_a^b$$

This method is easier conceptually but is usually more work.

2. Let $u = g(x)$ in the definite integral, change the <u>$x$-limits</u> $x = a$, $x = b$ to <u>u - limits</u> $u = g(a)$, $u = g(b)$

and evaluate

$$\int_{g(a)}^{g(b)} f(u)\, du.$$

This method is usually quicker, but you have to remember to change the limits of integration.

Ex 1 $\displaystyle\int_1^3 \frac{\ln x}{x}\,dx$

Method 1 Let $u = \ln x$

$$du = \frac{d}{dx}(\ln x)\,dx$$

$$= \frac{1}{x}\,dx$$

So $$\int \frac{\ln x}{x}\,dx = \int u\,du$$

$$= \frac{u^2}{2} + C$$

$$= \frac{(\ln x)^2}{2} + C.$$

Hence $\frac{(\ln x)^2}{2}$ is an antid of $\frac{\ln x}{x}$ and so by F.T.C

$$\int_1^3 \frac{\ln x}{x}\,dx = \left[\frac{(\ln x)^2}{2}\right]_1^3 = \frac{(\ln 3)^2}{2} - \frac{(\ln 1)^2}{2}$$

$$= \frac{(\ln 3)^2}{2} \quad (\text{as } \ln 1 = 0).$$

Method 2 Again let $u = \ln x$, but this time change limits.

when $x = 1$, $u = \ln 1 = 0$

$x = 3$, $u = \ln 3$.

Again $du = \frac{dx}{x}$, so

$$\int_1^3 \frac{\ln x}{x}\,dx = \int_0^{\ln 3} u\,du = \left[\frac{u^2}{2}\right]_0^{\ln 3}$$

$$= \frac{(\ln 3)^2}{2} - \frac{0^2}{2}$$

$$= \frac{(\ln 3)^2}{2} \quad \text{as before.}$$

__Ex 2__

$$\int_{-1}^{0} \frac{6x^2}{(2+x^3)^4}\,dx$$

__Method 2__ Let $u = 2 + x^3$, so

$$du = 3x^2\,dx$$

and $6x^2\,dx = 2du$.

__Limits__ When $x = -1$, $u = 2 + (-1)^3 = 1$

$x = 0$, $u = 2 + 0^3 = 2$.

Hence
$$\int_{-1}^{0} \frac{6x^2}{(2+x^3)^4}\,dx = \int_{1}^{2} \frac{2\,du}{u^4}$$

$$= 2\int_{1}^{2} u^{-4}\,du$$

$$= 2\left[\frac{u^{-3}}{-3}\right]_1^2$$

$$= 2\left[-\frac{1}{3u^3}\right]_1^2$$

$$= 2\left[-\frac{1}{3\times 8} - \left(-\frac{1}{3\times 1}\right)\right]$$

$$= 2\left(\frac{1}{3} - \frac{1}{24}\right)$$

$$= 2\left(\frac{8}{24} - \frac{1}{24}\right)$$

$$= \frac{2\times 7}{24} = \frac{14}{24} = \frac{7}{12}.$$

Method 1 ? Discuss in class.

Ex 3:

$$\int_1^3 \frac{dx}{5-x}$$

Let $u = 5-x$, $du = -dx$

$-du = dx$.

Limits: when $x = 1$, $u = 4$

$x = 3$, $u = 2$.

Get $$\int_4^2 -\frac{du}{u} = -\int_4^2 \frac{du}{u}$$

Limits wrong way round, but swapping the limits changes the sign of the integral and so allows us to get rid of the minus sign

$$= \int_2^4 \frac{du}{u} = \Big[\ln|u|\Big]_2^4$$

$$= \ln 4 - \ln 2$$

$$= \ln\left(\frac{4}{2}\right) = \ln 2 \approx \cdot 693.$$

Ex 4:

$$\int_0^{\ln 3} e^x (1+e^x)^{\frac{1}{2}}\, dx$$

Suggests $u = 1+e^x$

$du = e^x\, dx$

Limits When $x=0$, $u = 1+e^0 = 2$

$x = \ln 3$, $u = 1+e^{\ln 3}$

$= 1+3 = 4$

Integral then becomes

$$\int_2^4 u^{\frac{1}{2}}\, du = \left[\frac{2}{3}u^{3/2}\right]_2^4$$

$$= \left(\frac{2}{3}\cdot 4^{3/2} - \frac{2}{3}\, 2^{3/2}\right)$$

$$= \frac{2}{3}\left(8-\sqrt{8}\right) = \frac{4}{3}\left(4-\sqrt{2}\right).$$