MTH 142

SECOND SEMESTER CALCULUS

§ 5.3 Integration by Substitution

We make use of two ideas from first semester calculus

1. Antiderivatives

Recall that a fn F is an antiderivative (antid.) of another fn f if

$$F' = f$$

$$F \xrightarrow{\text{diff}} f$$

$$f \xrightarrow{\text{antidiff.}} F$$

2. The Chain Rule

$$\frac{d}{dx}\left(f(g(x))\right) = f'(g(x)) \cdot g'(x)$$

Hence if f, g are fns and f has antid F, then by the chain rule

$$\frac{d}{dx}\left(F(g(x))\right) = F'(g(x)) \cdot g'(x)$$

$$= f(g(x)) \cdot g'(x)$$

as F is an antid of f.

Thus, $F(g(x))$ is an antid. of $f(g(x)) \cdot g'(x)$.

If we recall that we can obtain all possible antiderivatives by adding an arbitrary constant to obtain the indefinite integral, then we can rephrase the above as

$$\int f(g(x)) \cdot g'(x)\,dx = F(g(x)) + C$$

where C is an arbitrary (unspecified) constant.

In practice, we denote $g(x)$ by the intermediate variable u and rewrite the derivative

$$\frac{du}{dx} = g'(x)$$

in the differential form

$$du = g'(x)\,dx$$

so that

$$\int f(g(x))\,g'(x)\,dx = \int f(u)\,du$$

$$= F(u) + C$$

$$= F(g(x)) + C$$

Note that $\int f(u)\,du$ is (hopefully) simpler than $\int f'(g(x))\,g'(x)\,dx$.

The tricky part, then is to find a choice of u which works.

The best thing to try is usually to look for a composite function in part of the integrand and then let u be the 'inside' function.

However, the composite function is often 'disguised' and we may need additional tricks such as algebra and trig identities to make things work out.

Ex 1 $\displaystyle\int (x^2+1)^{50} \cdot 2x \, dx$

Here $(x^2+1)^{50}$ is a composite function which suggests we try $u = x^2+1$.

Then $\dfrac{du}{dx} = 2x$, so that

$$du = 2x\,dx.$$

Hence we can rewrite the integral as

$2x\,dx$

$$\int u^{50} \; du$$

$(x^2+1)^{50}$

$$= \frac{u^{51}}{51} + C = \frac{(x^2+1)^{51}}{51} + C.$$

N.b. Never forget to convert back to x at the end!

Ex 2. $\int \cos(5x+3)\,dx$

Here $\cos(5x+3)$ is a composite f^n which suggests $u = 5x+3$.

Then $\frac{du}{dx} = 5$, so

$$du = 5\,dx.$$

However in the integrand we only have dx instead of $5\,dx$.

We get round this by dividing both sides of the eq^n for du by 5, i.e.

$$\frac{du}{5} = dx.$$

We can then rewrite the integral in terms of u as

$$\int \cos(u) \cdot \frac{du}{5} = \frac{1}{5} \int \cos u \, du$$

$$= \frac{\sin u}{5} + C$$

$$= \frac{\sin(5x+3)}{5} + C$$

This example illustrates two important points.

1. If the 'inside' fn is linear, we can let u be this linear function.

2. We can get away with being out by a multiplicative constant on the du (but we can't get away with anything worse than this).

Ex 3. $\displaystyle\int \frac{x}{x^2 - 7}\,dx$

Here it is not quite so obvious that we have a composite fn in our integrand, but if we look at

$$\frac{1}{x^2 - 7}$$

on its own, we see that this is a composite fn (where the outer fn is $1/x$) and so this suggests we let

$$u = x^2 + 7$$

Then $du = 2x\,dx$

and, dividing by 2 like before,

$$\frac{du}{2} = x\,dx$$

which matches what we have in the integral.

Substituting for u gives

$$\int \frac{x\,dx}{x^2-7} = \int \frac{du/2}{u}$$

$$= \frac{1}{2}\int \frac{du}{u}$$

$$= \frac{1}{2}\ln|u| + C$$

$$= \frac{1}{2}\ln|x^2-7| + C.$$

Ex 4. $\displaystyle\int x^5 \sqrt{1+x^2}\, dx$

Here the composite fn part of the integrand is $\sqrt{1+x^2}$ which suggests we try $u = 1+x^2$ (although it is not yet so clear what we're going to do with the rest of the integral!)

Anyway,

$$u = 1 + x^2$$

$$du = 2x\, dx$$

so $$\frac{du}{2} = x\, dx.$$

Since $x^5\, dx = x^4 \cdot x\, dx$, we have taken care of all the integrand except the x^4 bit.

For this we use algebra:

$$u = 1 + x^2$$

$$u - 1 = x^2,$$

So $x^4 = (u-1)^2$.

Rewriting in terms of u gives

$$\int \underbrace{(u-1)^2}_{x^4} \cdot \overbrace{\sqrt{u}}^{\sqrt{x^2+1}} \ \frac{du}{2} \qquad x\,dx$$

which we need to multiply out fully before we integrate

$$= \int (u^2 - 2u + 1) \cdot u^{\frac{1}{2}} \ \frac{du}{2}$$

$$= \frac{1}{2} \int (u^{5/2} - 2u^{3/2} + u^{1/2})\, du$$

$$= \frac{1}{2} \left(\frac{2}{7} u^{7/2} - 2 \times \frac{2}{5} u^{5/2} + \frac{2}{3} u^{3/2} \right) + C$$

$$= \frac{u^{7/2}}{7} - \frac{2u^{5/2}}{5} + \frac{u^{3/2}}{3} + C.$$

Don't forget to rewrite in terms of x!

$$= \frac{(1+x^2)^{7/2}}{7} - \frac{2(1+x^2)^{5/2}}{5} + \frac{(1+x^2)^{3/2}}{3} + C.$$

Ex 5 $\quad \int \sqrt{e^x}\, dx$

There are two main ways of solving this problem - both of them instructive.

1. Naïve method.

Let $u = e^x$ (inside fn).

Then $du = e^x \, dx$

$du = u \, dx$

So $\frac{du}{u} = dx$.

Rewriting in u:

$$\int \sqrt{u} \cdot \frac{du}{u} = \int u^{\frac{1}{2}} \frac{du}{u}$$

$$= \int u^{-\frac{1}{2}} du$$

$$= \frac{u^{\frac{1}{2}}}{\frac{1}{2}} + C$$

$$= 2\sqrt{u} + C$$

$$= 2\sqrt{e^x} + C.$$

2. Clever method.

We process the integrand a bit before making our substitution

$$\int \sqrt{e^x}\, dx = \int (e^x)^{\frac{1}{2}} dx$$

$$= \int e^{x/2}\, dx$$

which suggests we let

$$u = \frac{x}{2}$$

$$du = dx/2$$

$$dx = 2\,du$$

Rewrite:

$$\int \sqrt{e^x}\,dx = \int e^u \cdot 2\,du$$

$$= 2\int e^u\,du$$

$$= 2e^u + C$$

$$= 2e^{x/2} + C$$

$\left(= 2\sqrt{e^x} + C \right.$ as before $\left. \right)$

Ex 6 $\int \sin^3(x)\,dx$

Here is an example where letting u be the inside fn (ie $u = \sin x$) doesn't work – try it!

Instead we need to use a bit of algebra and the trig identity

$$\sin^2 x = 1 - \cos^2 x$$

before we make our substitution.

$$\int \sin^3 x\,dx = \int \sin^2 x \cdot \sin x\,dx$$

$$= \int (1 - \cos^2 x) \cdot \sin x\,dx$$

This new form suggests we try

$$u = \cos x$$

which does work.

$$du = -\sin x \, dx$$

$$-du = \sin x \, dx$$

So $$\int \sin^3 x \, dx = \int (1-u^2) \cdot -du$$

$$= \int (u^2 - 1) \, du$$

$$= \frac{u^3}{3} - u + C$$

$$= \frac{\cos^3 x}{3} - \cos x + C$$

Recap on Substitution

1. Let u be the 'inside' $f^{\underline{n}}$.

   Then $du = u'(x)\,dx$.

   Sometimes you need to first manipulate the integrand using algebra or trig in order to find a u which works.

2. Rewrite the integral as an integral in u.

3. Evaluate the integral in u.

4. Convert u back to x.

If at first you don't succeed, then try, try again!