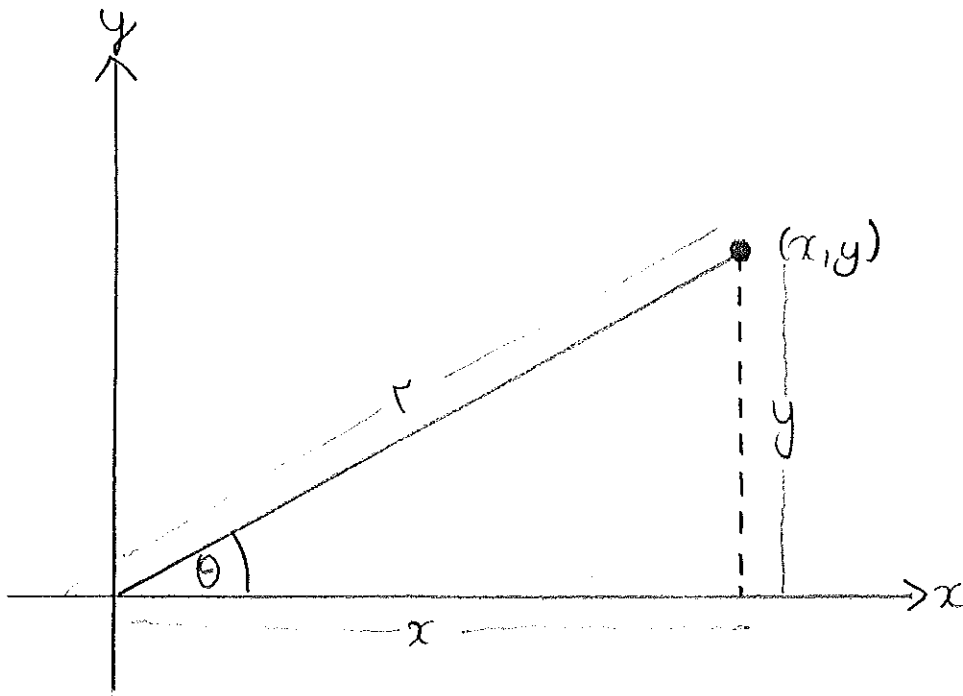


# § 10.2 § 10.3 Polar Coordinates



Conversion of polar coords. to Cartesian coords.

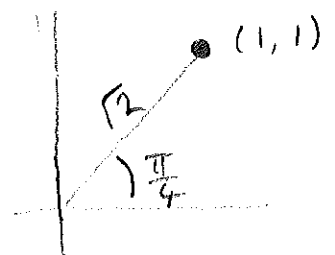
$$\frac{x}{r} = \cos \theta, \quad \frac{y}{r} = \sin \theta$$

$$\text{So } \boxed{x = r \cos \theta, \quad y = r \sin \theta}$$

e.g. If  $\theta = \frac{\pi}{4}, r = \sqrt{2}$

$$x = \sqrt{2} \cos \frac{\pi}{4} = \sqrt{2} \cdot \frac{1}{\sqrt{2}} = 1$$

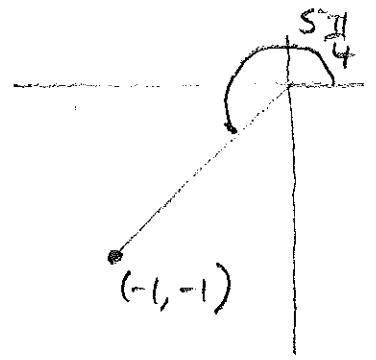
$$y = \sqrt{2} \sin \frac{\pi}{4} = \sqrt{2} \cdot \frac{1}{\sqrt{2}} = 1$$



$$\text{If } \theta = \frac{5\pi}{4}, r = \sqrt{2}$$

$$x = \sqrt{2} \cos\left(\frac{5\pi}{4}\right) = \sqrt{2} \cdot \frac{-1}{\sqrt{2}} = -1$$

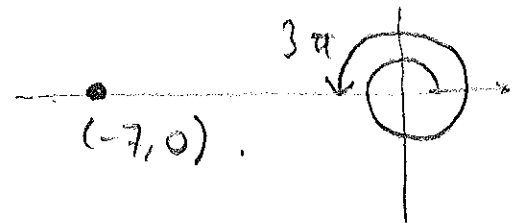
$$y = \sqrt{2} \sin\left(\frac{5\pi}{4}\right) = \sqrt{2} \cdot \frac{-1}{\sqrt{2}} = -1$$



$$\text{If } \theta = 3\pi, r = 7$$

$$x = 7 \cos(3\pi) = 7(-1) = -7$$

$$y = 7 \sin(3\pi) = 7(0) = 0$$



Conversion of Cartesian Coords to Polar Coords.

$$\frac{y}{x} = \tan\theta, x \neq 0 \quad x^2 + y^2 = r^2$$

$$\tan\theta = \frac{y}{x}, x \neq 0 \quad r^2 = x^2 + y^2$$

So

$$\theta = \arctan \frac{y}{x}, \quad r = \sqrt{x^2 + y^2}$$

Note that  $\theta = \arctan\left(\frac{y}{x}\right)$  is not sufficient to tell in which quadrant  $\theta$  should be.

e.g. If  $x=1, y=1$

$$\theta = \arctan\left(\frac{1}{1}\right) = \arctan 1 = \frac{\pi}{4} \quad \text{as we need a pt in the first quadrant}$$

However, if  $x=-1, y=-1$

$$\theta = \arctan\left(\frac{-1}{-1}\right) = \arctan 1 = \frac{5\pi}{4} \quad \text{as we need a pt in the third quadrant}$$

Note that there are many choices of  $\theta$  for a given  $(x, y)$ .

e.g. If  $x=-7, y=0$

$$\theta = \arctan(0)$$

$\theta = \pi$  will do as it's on the -ve x-axis

However, so would  $\theta = -\pi$ ,  $\theta = 3\pi$ ,  $\theta = 5\pi$  etc.

In fact, all possible values of  $\theta$  are

$$\theta = \pi + 2n\pi, \quad \text{where } n \text{ is any integer}$$

What if  $x = 0$ ?

Here  $\theta = \arcsin\left(\frac{y}{x}\right)$  doesn't make sense.

However, if  $y > 0$ , then we need a value of  $\theta$  which puts us on the positive  $y$ -axis (e.g.  $\theta = \frac{\pi}{2}$ ,  $\theta = \frac{5\pi}{2}$ ).

If  $y < 0$ , then we need a value of  $\theta$  which puts us on the negative  $y$ -axis (e.g.  $\theta = -\frac{\pi}{2}$ ,  $\theta = \frac{3\pi}{2}$ ).

Finally if  $x = y = 0$  and we're at the origin, then  $r = 0$  and  $\theta$  is undefined.

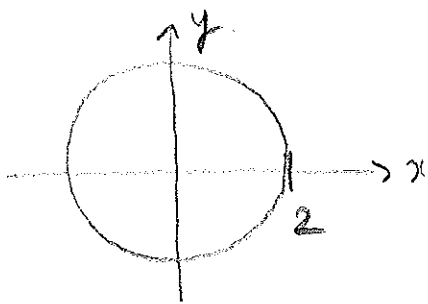
Except for this case, given  $(x, y)$  all possible values of  $\theta$  differ by integer multiples of  $2\pi$  like in the example above.

# Curves in Polar Coordinates

i)  $r = 2$

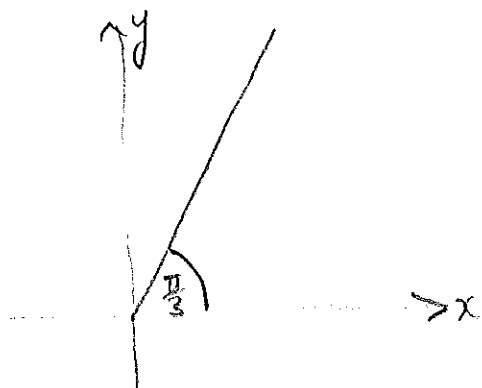
This is just a circle of radius 2 about the origin since

$$r = 2 \Rightarrow \sqrt{x^2 + y^2} = 2 \quad x^2 + y^2 = 2^2$$



ii)  $\theta = \frac{\pi}{3}$

This is a half-line with slope  $\tan \frac{\pi}{3} = \sqrt{3}$  which extends from the origin.

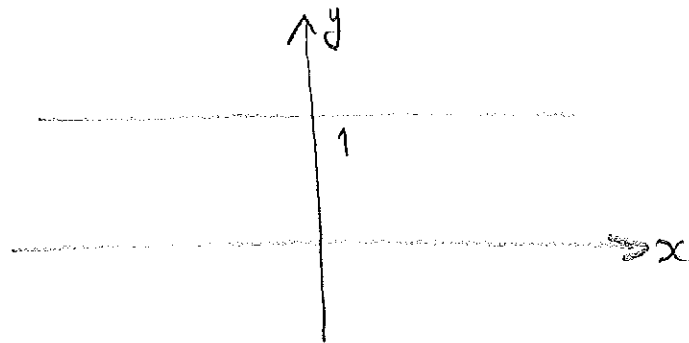


iii)  $y = 1$ .

This is a horizontal line in Cartesian coordinates. Since  $y = r \sin \theta$ ,

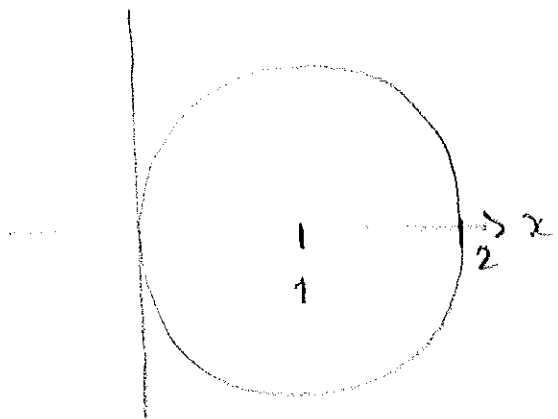
$$r \sin \theta = 1$$

$$r = \frac{1}{\sin \theta}$$



iv)  $r = 2 \cos \theta$

This is a circle of radius 1 centered at (1, 0).



To see this, remember that  $\cos \theta = \frac{x}{r}$ , so

$$r = 2 \cos \theta = \frac{2x}{r}$$

$$r^2 = 2x$$

$$x^2 + y^2 = 2x \quad \text{as} \quad x^2 + y^2 = r^2$$

Then

$$x^2 - 2x + y^2 = 0$$

Complete the square

$$x^2 + 2(-1)x + y^2 = 0$$

$$x^2 + 2(-1)x + (-1)^2 - (-1)^2 + y^2 = 0$$

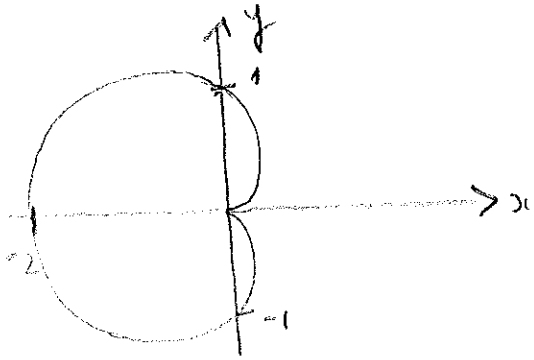
$$(x - 1)^2 - 1 + y^2 = 0$$

$$(x - 1)^2 + y^2 = 1$$

which is indeed the eqn. of a circle of radius 1 with centre  $(1, 0)$ .

$$v) \quad r = 1 - \cos \theta$$

This curve is called a cardioid



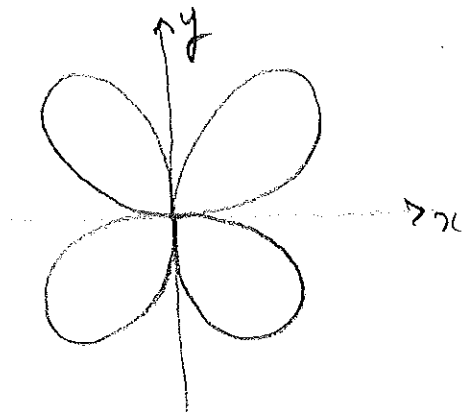
More generally, curves of the form

$$r = a + b \sin \theta \quad \text{or} \quad r = a + b \cos \theta$$

are called limaçons.

$$vi) \quad r = 3 \sin 2\theta$$

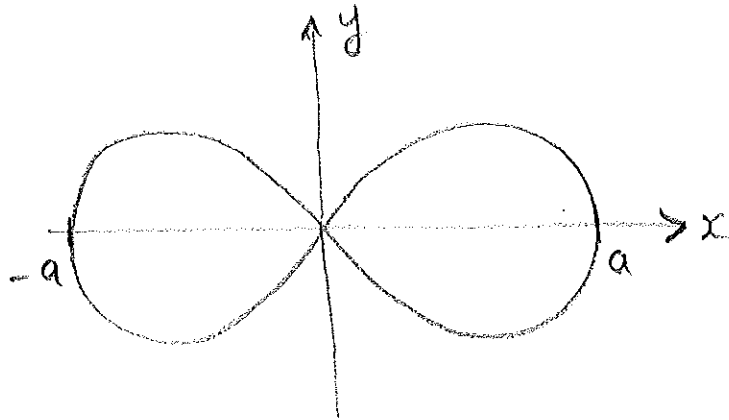
This curve is called a four-leaved rose





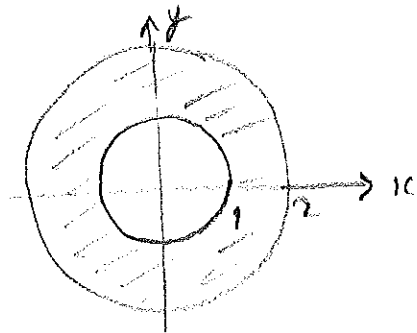
vii)  $r^2 = a^2 \cos 2\theta$  ( $r=0$  if  $\cos 2\theta < 0$ )

This curve is called a lemniscate



### Regions in Polar Coordinates

i).  $1 \leq r \leq 2$

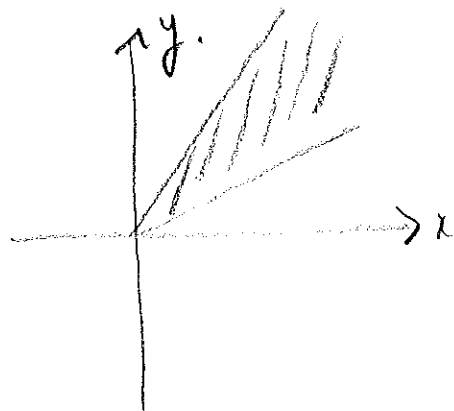


Annulus (ring) of  
inner radius 1 and  
outer radius 2.

ii)  $\frac{\pi}{6} \leq \theta \leq \frac{\pi}{3}$

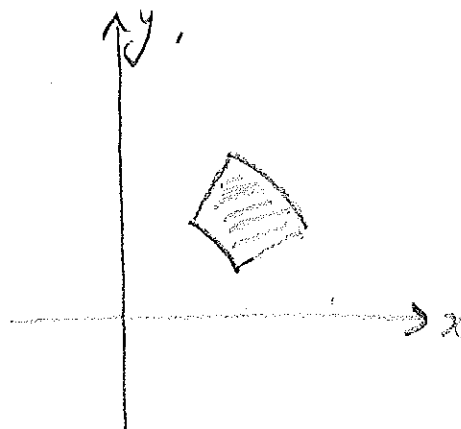
Sector of angle  $\frac{\pi}{6}$  ( $30^\circ$ )

between  $\frac{\pi}{6}$  and  $\frac{\pi}{3}$



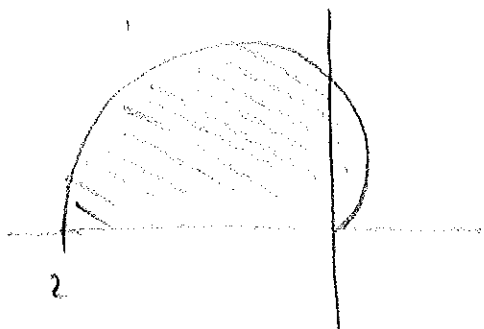
iii)  $1 \leq r \leq 2, \frac{\pi}{6} \leq \theta \leq \frac{\pi}{3}$

Curry rectangle



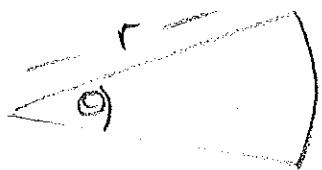
iv)  $0 \leq \theta \leq \pi, r \leq 1 - \cos \theta$

Interior of half a cardioid



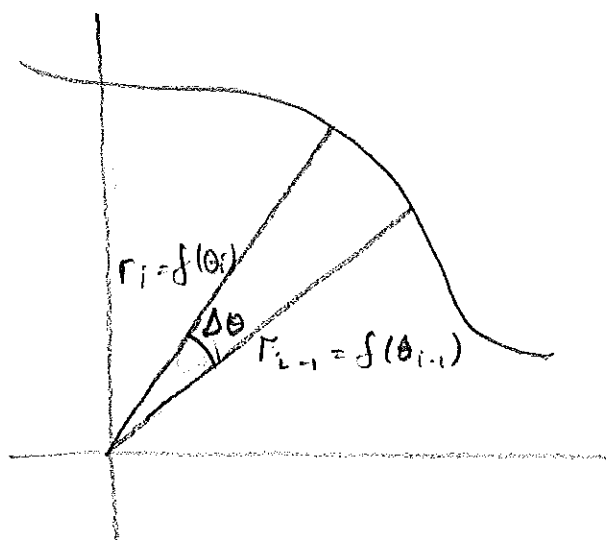
# Area in Polar Coordinates

Recall that the area of a sector of radius  $r$  and angle  $\theta$  is  $\frac{1}{2} r^2 \theta$ .



Suppose now we have a polar graph

$$r = f(\theta), \quad \alpha \leq \theta \leq \beta \quad (\beta - \alpha \leq 2\pi)$$



The area of the small part of the graph is approx that of a sector of radius

$$r_{i-1} = f(\theta_{i-1})$$

and angle  $\Delta\theta$  while

the area of such a sector is

$$\frac{1}{2} r_{i-1}^2 \Delta\theta = \frac{(f(\theta_{i-1}))^2}{2} \Delta\theta.$$

Add these contributions together to get an approximation to the total area  $A$

$$A \approx \frac{1}{2} \sum_{j=1}^n (f(\theta_{j-1}))^2 \Delta \theta$$

$$= \frac{1}{2} \sum_{i=0}^{n-1} (f(\theta_i))^2 \Delta \theta$$

(let  $i = j - 1$ )

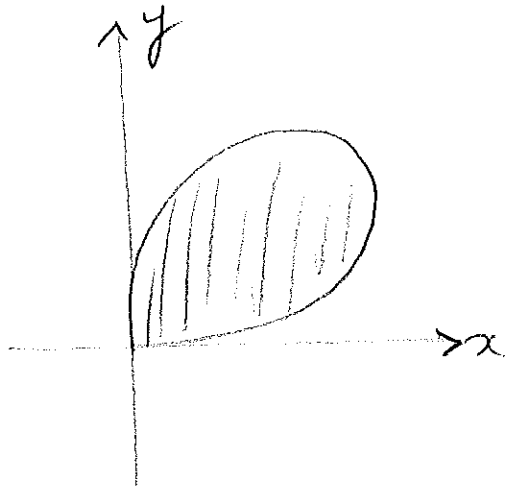
as  $j$  goes from 1 to  $n$   
 $i$  goes from 0 to  $n-1$ )

This Riemann sum becomes an integral as we let  $n \rightarrow \infty$  and we define

$$A := \frac{1}{2} \int_{\alpha}^{\beta} (f(\theta))^2 d\theta.$$

$$= \frac{1}{2} \int_{\alpha}^{\beta} r^2(\theta) d\theta$$

Ex. Find the area of one petal of the four-leaved rose  $r = 3\sin 2\theta$ .



Here  $0 \leq \theta \leq \frac{\pi}{2}$  and

$$A = \frac{1}{2} \int_0^{\frac{\pi}{2}} r^2(\theta) d\theta$$

$$= \frac{1}{2} \int_0^{\frac{\pi}{2}} 9 \sin^2(2\theta) d\theta$$

$$= \frac{1}{2} \int_0^{\frac{\pi}{2}} 9 \cdot \frac{1}{2} (1 - \cos(4\theta)) d\theta$$

using  $\sin^2 x = \frac{1}{2} (1 - \cos 2x)$

and  $4\theta = 2(2\theta)$ .

$$= \frac{9}{4} \int_0^{\frac{\pi}{2}} (1 - \cos 4\theta) d\theta$$

$$= \frac{9}{4} \left[ \theta - \frac{\sin 4\theta}{4} \right]_0^{\frac{\pi}{2}}$$

$$= \frac{9}{4} \left( \left( \frac{\pi}{2} - \frac{\sin 2\pi}{4} \right) - \left( 0 - \frac{\sin 0}{4} \right) \right)$$

$$= \frac{9}{4} \left( \frac{\pi}{2} - 0 - 0 \right)$$

$$= \frac{9\pi}{8}$$

## Slope in Polar Coordinates

A polar curve  $r = f(\theta)$  can be expressed as a parametric Cartesian curve with parameter  $\theta$ , since

$$x = r \cos \theta = f(\theta) \cos \theta$$

$$y = r \sin \theta = f(\theta) \sin \theta.$$

If  $y$  is a fn of  $x$ , then by the Leibniz form of the chain rule

$$\frac{dy}{d\theta} = \frac{dy}{dx} \cdot \frac{dx}{d\theta}$$

and so if  $\frac{dx}{d\theta} \neq 0$ , we can divide by  $\frac{dx}{d\theta}$  to get

$$\frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{dy}{dx}, \quad \text{or} \quad \frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}$$

Ex. Find the slope of the curve

$$y = 3 \sin 2\theta \quad \text{at} \quad \theta = \frac{\pi}{3}.$$

$$x = r \cos \theta = 3 \sin 2\theta \cos \theta$$

so  $\frac{dx}{d\theta} = 6 \cos 2\theta \cos \theta - 3 \sin 2\theta \sin \theta$  by product rule.

Similarly  $y = r \sin \theta = 3 \sin 2\theta \sin \theta$

$$\frac{dy}{d\theta} = 6 \cos 2\theta \sin \theta + 3 \sin 2\theta \cos \theta$$

Thus

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{6 \cos 2\theta \sin \theta + 3 \sin 2\theta \cos \theta}{6 \cos 2\theta \cos \theta - 3 \sin 2\theta \sin \theta}$$

When  $\theta = \frac{\pi}{3}$   $\cos \theta = \frac{1}{2}$ ,  $\sin \theta = \frac{\sqrt{3}}{2}$

$2\theta = \frac{2\pi}{3}$   $\cos 2\theta = -\frac{1}{2}$ ,  $\sin 2\theta = \frac{\sqrt{3}}{2}$ .



$$\frac{dy}{dx} = \frac{6(-\frac{1}{2})(\frac{\sqrt{3}}{2}) + 3(\frac{\sqrt{3}}{2})(\frac{1}{2})}{6(-\frac{1}{2})(\frac{1}{2}) - 3(\frac{\sqrt{3}}{2})(\frac{\sqrt{3}}{2})}$$

$$= \frac{\frac{\sqrt{3}}{2}(-3 + \frac{3}{2})}{6(-\frac{1}{4}) - 3(\frac{3}{4})}$$

$$= \frac{\frac{\sqrt{3}}{2}(-\frac{3}{2})}{- \frac{6}{4} - \frac{9}{4}}$$

$$= \frac{\frac{\sqrt{3}}{2} \cdot \frac{3}{2}}{\frac{15}{4}}$$

$$= \frac{3\sqrt{3}}{15}$$

$$= \frac{\sqrt{3}}{5}$$

## Arc Length in Polar Coordinates

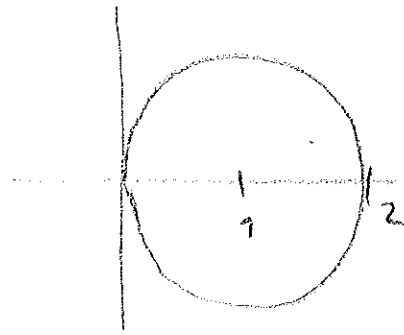
Since for  $r = f(\theta)$ ,  $\alpha \leq \theta \leq \beta$   
we have the parametric form

$$x = f(\theta) \cos \theta, \quad y = f(\theta) \sin \theta,$$

the arc length is given by the usual  
formula for a parametric curve.

$$S = \int_{\alpha}^{\beta} \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2} d\theta$$

Ex.  $r = 2 \cos \theta$



Here  $x = 2 \cos \theta \cos \theta = 2 \cos^2 \theta$

$$y = 2 \cos \theta \sin \theta$$

$$\frac{dx}{d\theta} = -4 \cos \theta \sin \theta = -2 \sin 2\theta$$

as  $\sin 2\theta = 2 \sin \theta \cos \theta$

$$\frac{dy}{d\theta} = -2 \sin^2 \theta + 2 \cos^2 \theta = 2 \cos 2\theta$$

as  $\cos 2\theta = \cos^2 \theta - \sin^2 \theta$

$$\sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}$$

$$= \sqrt{4 \sin^2 2\theta + 4 \cos^2 2\theta}$$

$$= \sqrt{4(\sin^2 \theta + \cos^2 \theta)}$$

$$= \sqrt{4}$$

as  $\cos^2 \theta + \sin^2 \theta = 1$

$$= 2$$

$S_0$

$$S = \int_0^{2\pi} \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2} d\theta$$

$$= \int_0^{2\pi} 2 d\theta$$

$$= [2\theta]_0^{2\pi}$$

$$= 4\pi$$