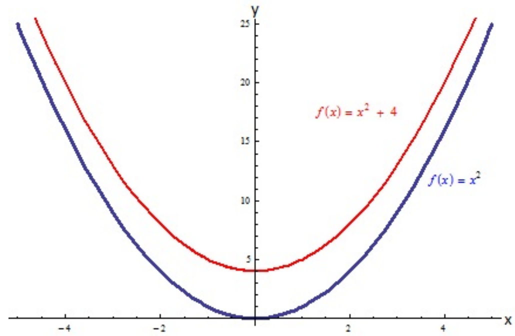
Idea:

Build up more complicated functions from simpler ones- like a lego.

SHIFTS:

Consider $f(x) = x^2$.

Vertical Shift



Original

$y = x^2$

Replace y by $y - 4$

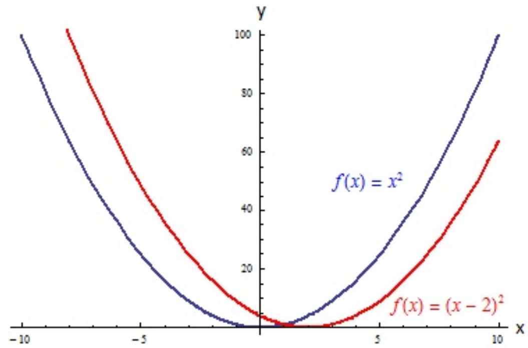
$y - 4 = x^2$

New

$y = x^2 + 4$

Replacing y by $y - 4$ move graph vertically up by 4.

Horizontal Shift



Original

$y = x^2$

Replace x by $x - 2$

$y = (x - 2)^2$

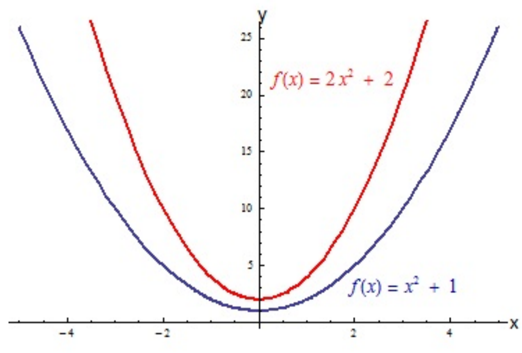
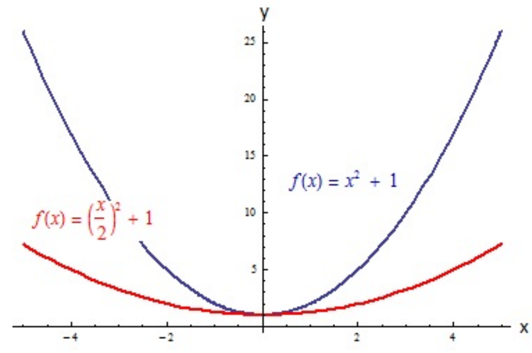
New

$y = (x - 2)^2 = x^2 - 4x + 4$

Replacing x by $x - 2$ moves graph to the right by 2.

Stretches:

Consider $g(x) = x^2 + 1$.

Vertical Stretch	Horizontal Stretch
	
<u>Original</u> $y = x^2 + 1$ Replace y by $y/2$ $y/2 = x^2 + 1$ <u>New</u> $y = 2x^2 + 2$ Replacing y by $y/2$ stretches graph vertically by a factor of 2	<u>Original</u> $y = x^2 + 1$ Replace x by $x/2$ $y = (x/2)^2 + 1$ <u>New</u> $y = \frac{x^2}{4} + 1$ Replacing x by $x/2$ stretches graph horizontally by a factor of 2.

MORAL:
For a function $f(x)$

Shifts:

- Replacing y by $y - k$ moves the graph of f up by k.
- Replacing x by $x - k$ moves the graph of f to the right by k.

Stretches:

- Replacing y by y/k stretches the graph of f vertically by a factor of k.
- Replacing x by x/k stretches the graph of f horizontally by a factor of k.

Composite Functions:

Oil is spilled from a tanker. The slick, which is always a perfect circle, grows with time. The area, A, of the oil slick is a function of it's radius r,

$$A = f(r) = \pi r^2$$

The radius of the slick increases as time passes, so the radius, r, is a function of the time, t. If, for example, the radius is given by

$$r = g(t) = 1 + t$$

then the area as a function of time is given by substitution,

$$A = \pi r^2 = \pi(1 + t)^2$$

We are thinking of A as a composite function or a 'function of a function', which is written

$$A = f(g(t)) = \pi(g(t))^2 = \pi(1 + t)^2$$

Here we do g first then do f.

Example 1:

If $f(x) = x^2$ and $g(x) = x + 1$ find

$$\text{a) } f(g(2)),\ \text{b) } g(f(2)),\ \text{c) } f(g(x)),\ \text{d) } g(f(x))$$

a)

$$\begin{aligned} f(g(2)) &= f(2 + 1) \\ &= f(3) \\ &= 3^2 \\ &= 9 \end{aligned}$$

b)

$$\begin{aligned} g(f(2)) &= g(2^2) \\ &= g(4) \\ &= 4 + 1 \\ &= 5 \end{aligned}$$

NOTE THAT: $f(g(2)) \neq g(f(2))$

c)

$$\begin{aligned} f(g(x)) &= f(x+1) \\ &= (x+1)^2 \\ &= x^2 + 2x + 1 \end{aligned}$$

d)

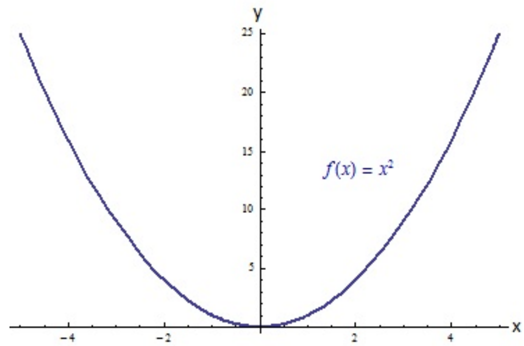
$$\begin{aligned} g(f(x)) &= g(x^2) \\ &= x^2 + 1 \end{aligned}$$

AGAIN NOTE THAT: $f(g(x)) \neq g(f(x))$
The point of this example is to show that the **order of composition matters.**

NOTATION:
Sometimes instead of $f(g(x))$, it is written $(f \circ g)(x)$.

Odd and Even Functions, Symmetry

Consider $f(x) = x^2$.

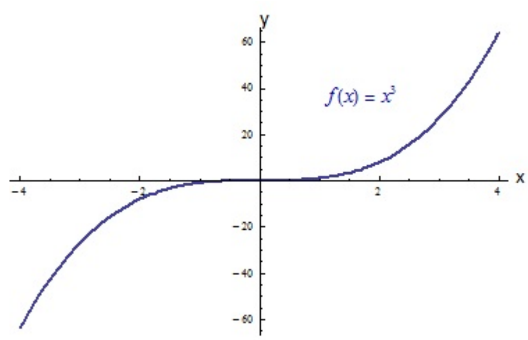


Here

$$\begin{aligned} f(-x) &= (-x)^2 \\ &= x^2 \\ &= f(x) \end{aligned}$$

This means the graph of f is symmetric under reflection in the y-axis. Such a function is called even.

Now consider $g(x) = x^3$.



Here

$$\begin{aligned} g(-x) &= (-x)^3 \\ &= -x^3 \\ &= -g(x) \end{aligned}$$

This means the graph of g is symmetric under 180 deg rotation about $(0, 0)$. Such a function is called odd.

Def'n: For any function, f,
f is even if $f(-x) = f(x)$ for all x
f is odd if $f(-x) = -f(x)$ for all x.

Question:
What can we say about $g(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2}$, $h(x) = x^{\frac{1}{3}}$?

Question:
If f is odd, can we say anything about $f(0)$, assuming it exists?

Inverse Functions:

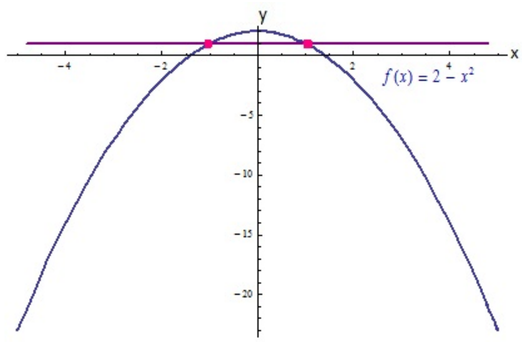
Consider $f(x) = x^3$.

Set $y = x^3$, $y^{\frac{1}{3}} = x$, or $x = y^{\frac{1}{3}}$.

Hence, in this example, given y as a function of x, we were able, by manipulation, to get x as a function of

y. This is the basic idea of inverse functions.

Now consider $g(x) = 2 - x^2$.



If we set $y = 1$.

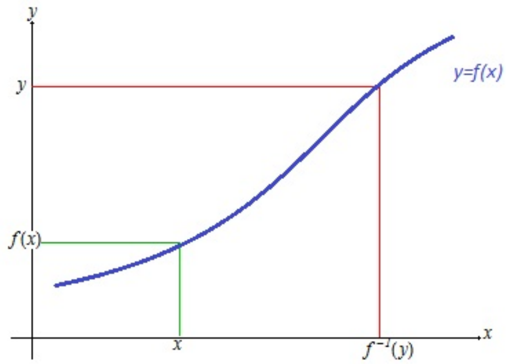
$$
\begin{aligned}
1 &= 2 - x^2 \\
1 + x^2 &= 2 \\
x^2 &= 2 - 1 \\
x^2 &= 1 \\
x &= \pm 1
\end{aligned}
$$

There are two values of x to choose from (see graph). In this case, we have no hope of getting x as a function of y.

If a function has an inverse, we say it is invertible.

Fact:

A function is invertible if and only if its graph intersects any horizontal line at most once (call the **horizontal line test**).



Fact:

If f is either increasing or decreasing, then it is invertible since it passes the horizontal line test.

Def'n:

If f is invertible, its inverse, f^{-1} defined as follows:

$$f^{-1}(y) = x$$

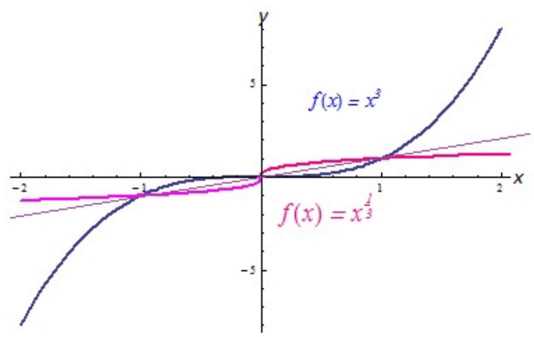
means $y = f(x)$.

In practice, we set $y = f(x)$ an solve for x as we did in the first example.

Graphs of Inverse Functions:

Consider (again) $f(x) = x^3$. Setting $y = x^3$ and solving we had $x = y^{\frac{1}{3}}$, or $f^{-1}(y) = y^{\frac{1}{3}}$, or if we want to change our independent variable back to x, $f^{-1}(x) = x^{\frac{1}{3}}$.

Now the point $(2, 8)$ is on the graph of f, as $8 = 2^3$. Similarly, the point $(8, 2)$ is on the graph of f^{-1}. $(8, 2)$ is obtained from $(2, 8)$ by swapping the coordinates which is equivalent to reflecting in the line $y = x$.



In general, for any invertible function f, the graph of f^{-1} is obtain from that of f by reflecting in the line $y = x$.