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What is $\mathbb{R}^n$?

**Notation and Terminology**

- $\mathbb{R}$ denotes the set of real numbers.
- $\mathbb{R}^2$ denotes the set of all column vectors with two entries.
- $\mathbb{R}^3$ denotes the set of all column vectors with three entries.
- In general, $\mathbb{R}^n$ denotes the set of all column vectors with $n$ entries.

**Scalar quantities versus vector quantities**

- A **scalar** quantity has only magnitude; e.g. time, temperature.
- A **vector** quantity has both magnitude and direction; e.g. displacement, force, wind velocity.

Whereas two scalar quantities are equal if they are represented by the same value, two vector quantities are equal if and only if they have the same magnitude and direction.
Vectors in $\mathbb{R}^2$ and $\mathbb{R}^3$ have convenient geometric representations as position vectors of points in the 2-dimensional (Cartesian) plane and in 3-dimensional space, respectively.
Notation

- If $P$ is a point in $\mathbb{R}^n$ with coordinates $(p_1, p_2, ..., p_n)$ we denote this by $P = (p_1, p_2, ..., p_n)$.
- If $P = (p_1, p_2, ..., p_n)$ is a point in $\mathbb{R}^n$, then
  $$\overrightarrow{P} = \begin{bmatrix} p_1 \\ p_2 \\ \vdots \\ p_n \end{bmatrix}$$

  is often used to denote the position vector of the point.

- Instead of using a capital letter to denote the vector (as we generally do with matrices), we emphasize the importance of the geometry and the direction with an arrow over the name of the vector.
Notation and Terminology

- The notation $\vec{P}$ emphasizes that this vector goes from the origin 0 to the point $P$. We can also use lower case letters for names of vectors. In this case, we write $\vec{P} = \vec{p}$.

- Any vector

  $$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

  is associated with the point $(x_1, x_2, \ldots, x_n)$. Please notice that in some context $\vec{x}$ can be a row vector or a column vector.

- Often, there is no distinction made between the vector $\vec{x}$ and the point $(x_1, x_2, \ldots, x_n)$, and we say that both $(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ and $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^n$. 

Jonathan Chávez
**Algebra in \( \mathbb{R}^n \)**

**Addition in \( \mathbb{R}^n \)**

Since vectors in \( \mathbb{R}^n \) are \( n \times 1 \) matrices, addition in \( \mathbb{R}^n \) is precisely matrix addition using column or row matrices, i.e.,

- If \( \vec{u} \) and \( \vec{v} \) are in \( \mathbb{R}^n \), then \( \vec{u} + \vec{v} \) is obtained by adding together corresponding entries of the vectors.
- The zero vector in \( \mathbb{R}^n \) is the \( n \times 1 \) zero matrix, and is denoted \( \vec{0} \).

**Example**

Let \( \vec{u} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \) and \( \vec{v} = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} \). Then,

\[
\vec{u} + \vec{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 5 \\ 7 \\ 9 \end{bmatrix}
\]
Properties of Vector Addition

Let \( \mathbf{u}, \mathbf{v}, \) and \( \mathbf{w} \) be vectors in \( \mathbb{R}^n \). Then the following properties hold.

1. \( \mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u} \) (vector addition is commutative).
2. \( (\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w}) \) (vector addition is associative).
3. \( \mathbf{u} + \mathbf{0} = \mathbf{u} \) (existence of an additive identity).
4. \( \mathbf{u} + (-\mathbf{u}) = \mathbf{0} \) (existence of an additive inverse).
Scalar Multiplication

Since vectors in $\mathbb{R}^n$ are $n \times 1$ matrices, scalar multiplication in $\mathbb{R}^n$ is precisely matrix scalar multiplication using column matrices, i.e., If $\overline{u}$ is a vector in $\mathbb{R}^n$ and $k \in \mathbb{R}$ is a scalar, then $k \overline{u}$ is obtained by multiplying every entry of $\overline{u}$ by $k$.

Example

Let $\overline{u} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ and $k = 4$. Then,

$$k \overline{u} = 4 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 4 \\ 8 \\ 12 \end{bmatrix}$$
Properties of Scalar Multiplication

Let \( \vec{u}, \vec{v} \in \mathbb{R}^n \) be vectors and \( k, p \in \mathbb{R} \) be scalars. Then the following properties hold.

1. \( k(\vec{u} + \vec{v}) = k\vec{u} + k\vec{v} \) (scalar multiplication distributes over vector addition).
2. \( (k + p)\vec{u} = k\vec{u} + p\vec{u} \) (addition distributes over scalar multiplication).
3. \( k(p\vec{u}) = (kp)\vec{u} \) (scalar multiplication is associative).
4. \( 1\vec{u} = \vec{u} \) (existence of a multiplicative identity).
Length of a Vector, $\mathbb{R}^2$

If $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{R}^2$, then the length of the vector $\mathbf{x}$ is the distance from the origin 0 to the point $X = (x_1, x_2)$ given by $d(0, X)$.

The length of $\mathbf{x}$, denoted $||\mathbf{x}||$, is given by:

$$d(0, X) = ||\mathbf{x}|| = \sqrt{\mathbf{x}^T \mathbf{x}} = \sqrt{x_1^2 + x_2^2}$$
Length of a Vector, $\mathbb{R}^n$

This extends clearly to $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^n$.

The length of $\mathbf{x}$ is the distance from the origin 0 to the point $X = (x_1, x_2, \ldots, x_n)$ given by $d(0, X)$.

$$d(0, X) = ||\mathbf{x}|| = \sqrt{\mathbf{x}^T \mathbf{x}} = \sqrt{x_1^2 + x_2^2 + \ldots + x_n^2}.$$

Please notice that if we define $\mathbf{x} = [x_1, x_2, \ldots, x_n]$ as a row vector. Then,

$$d(0, X) = ||\mathbf{x}|| = \sqrt{\mathbf{x} \mathbf{x}^T} = \sqrt{x_1^2 + x_2^2 + \ldots + x_n^2}.$$
Unit Vectors

Definition

A unit vector is a vector of length one.

Example

\[
\begin{bmatrix}
1 \\
0 \\
0
\end{bmatrix}, \begin{bmatrix}
0 \\
1 \\
0
\end{bmatrix}, \begin{bmatrix}
0 \\
0 \\
1
\end{bmatrix}, \begin{bmatrix}
\frac{\sqrt{2}}{2} \\
0 \\
\frac{\sqrt{2}}{2}
\end{bmatrix},
\]

are examples of unit vectors.

Example

If \( \vec{v} \neq \vec{0} \), then
\[
\frac{1}{||\vec{v}||} \vec{v}
\]

is a unit vector in the same direction as \( \vec{v} \).
Example

$$\mathbf{v} = \begin{bmatrix} -1 \\ 3 \\ 2 \end{bmatrix}$$ is not a unit vector, since $$||\mathbf{v}|| = \sqrt{14}$$. However,

$$\mathbf{u} = \frac{1}{\sqrt{14}} \mathbf{v} = \begin{bmatrix} \frac{-1}{\sqrt{14}} \\ \frac{3}{\sqrt{14}} \\ \frac{2}{\sqrt{14}} \end{bmatrix}$$

is a unit vector in the same direction as $$\mathbf{v}$$, i.e.,

$$||\mathbf{u}|| = \frac{1}{\sqrt{14}} ||\mathbf{v}|| = \frac{1}{\sqrt{14}} \sqrt{14} = 1.$$ 

Example

If $$\mathbf{v}$$ and $$\mathbf{w}$$ are nonzero that have

- the same direction, then $$\mathbf{v} = \frac{||\mathbf{v}||}{||\mathbf{w}||} \mathbf{w}$$;
- opposite directions, then $$\mathbf{v} = -\frac{||\mathbf{v}||}{||\mathbf{w}||} \mathbf{w}.$$
Definition

If

\[ \vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}, \text{ and } \vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} \]

are in \( \mathbb{R}^n \), then the dot product \( \vec{u} \cdot \vec{v} \) is as the \( 1 \times 1 \) matrix

\[ \vec{u}^T \vec{v} = [u_1, u_2, \ldots, u_n] \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = \begin{bmatrix} u_1 v_1 + u_2 v_2 + \cdots + u_n v_n \end{bmatrix} \]

which is treated as a scalar given by \( u_1 v_1 + u_2 v_2 + \cdots + u_n v_n \)

Please notice that this definition can be adapted if \( \vec{x} \) and \( \vec{y} \) are regarded as row vectors. The only change is that \( \vec{x} \cdot \vec{y} = \vec{x}^T \vec{y} \).
The Dot Product

Problem

Find $\vec{u} \cdot \vec{v}$ for $\vec{u} = [1, 2, 0, -1]^T$, $\vec{v} = [0, 1, 2, 3]^T$.

Solution

$$\vec{u} \cdot \vec{v} = (1)(0) + (2)(1) + (0)(2) + (-1)(3)$$
$$= 0 + 2 + 0 - 3 = -1$$
**Definition**

Let \( \vec{v}_1, \vec{v}_1, \ldots, \vec{v}_k \) be \( k \) vectors on \( \mathbb{R}^n \). A vector \( \vec{w} \in \mathbb{R}^n \) is said to be a **linear combination** of the vectors \( \vec{v}_1, \vec{v}_1, \ldots, \vec{v}_k \) if there exist constants \( a_1, a_2, \ldots, a_k \) (called coefficients) such that

\[
\vec{w} = \sum_{i=1}^{k} a_i \vec{v}_i = a_1 \vec{v}_1 + a_2 \vec{v}_2 + \ldots + a_k \vec{v}_k
\]

**Definition**

Let \( S = \{ \vec{v}_1, \vec{v}_1, \ldots, \vec{v}_k \} \) be a set of \( k \) vectors on \( \mathbb{R}^n \). That is, \( S \subset \mathbb{R}^n \).

The span of \( S \), written as \( \text{span}(S) \) is the **set of all linear combinations** of the elements of \( S \). That is,

\[
\text{span}(S) = \left\{ \sum_{i=1}^{k} a_i \vec{v}_i \text{ such that } a_1, a_2 \ldots a_k \in \mathbb{R} \right\}
\]

Please notice that for creating the span, we consider **ALL possible combinations** of the coefficients \( a_1, a_2, \ldots, a_k \).
Definition

A set of non-zero vectors \( \{ \vec{u}_1, \cdots, \vec{u}_k \} \) in \( \mathbb{R}^n \) is said to be \textbf{linearly independent} if whenever

\[
\sum_{i=1}^{k} a_i \vec{u}_i = \vec{0}
\]

it follows that each \( a_i = 0 \). A set that is \textbf{not} linearly independent is called \textbf{linearly dependent}.

We can rewrite the definition of linear independence as follows:

A set of non-zero vectors \( \{ \vec{u}_1, \cdots, \vec{u}_k \} \) in \( \mathbb{R}^n \) is said to be \textbf{linearly independent} if whenever \( \vec{0} \) is a linear combination of them, the coefficients of the linear combination are all 0.
A Linearly Dependent Set

Problem

Consider the vectors \( \vec{u} = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, \vec{v} = \begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix}, \vec{w} = \begin{bmatrix} 0 \\ 4 \\ 1 \end{bmatrix} \). Is the set \( \{\vec{u}, \vec{v}, \vec{w}\} \) linearly independent?

Solution

Notice that we can write \( \vec{w} \) as a linear combination of \( \vec{u}, \vec{v} \) as follows:

\[
\begin{bmatrix} 0 \\ 4 \\ 1 \end{bmatrix} = (-10) \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} + (7) \begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix}
\]

Hence, \( \vec{w} \) is in \( \text{span}\{\vec{u}, \vec{v}\} \). By the definition, this set is not linearly independent (it is linearly dependent).
Example

Is \( S = \left\{ \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} \right\} \) linearly independent?
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Solution
Problem

Let \( \{ \vec{u}, \vec{v}, \vec{w} \} \) be an independent set of \( \mathbb{R}^n \). Is \( \{ \vec{u} + \vec{v}, 2\vec{u} + \vec{w}, \vec{v} - 5\vec{w} \} \) linearly independent?

Solution
Problem

Describe the span of the vectors \( \vec{u} = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \) and \( \vec{v} = \begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix} \).
Problem

Let \([1, 1, 0]^T\) and \(\vec{v} = [3, 2, 0]^T \in \mathbb{R}^3\). Show that \(\vec{w} = [4, 5, 0]^T\) is in span \(\{\vec{u}, \vec{v}\}\).

Solution

For a vector to be in span \(\{\vec{u}, \vec{v}\}\), it must be a linear combination of these vectors. If \(\vec{w} \in \text{span} \{\vec{u}, \vec{v}\}\), we must be able to find scalars \(a, b\) such that
Problem

Let $[1, 1, 1]^T$ and $\vec{v} = [3, 2, 0]^T \in \mathbb{R}^3$. Does $\vec{w} = [4, 5, 0]^T$ belong to span $\{\vec{u}, \vec{v}\}$?

*This is almost identical to the previous, except that $\vec{u}$ (above) has one entry that is different.*

Solution

In this case, the system of linear equations is inconsistent which you can verify. Therefore $\vec{w} \notin \text{span} \{\vec{u}, \vec{v}\}$. 
Definition

Let $\vec{e}_j$ denote the $j^{th}$ column of $I_n$, the $n \times n$ identity matrix; $\vec{e}_j$ is called the $j^{th}$ coordinate vector of $\mathbb{R}^n$.

Claim

$\mathbb{R}^n = \text{span}\{\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n\}$.

Proof.

Let $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^n$. Then $\vec{x} = x_1 \vec{e}_1 + x_2 \vec{e}_2 + \cdots + x_n \vec{e}_n$, where $x_1, x_2, \ldots, x_n \in \mathbb{R}$. Therefore, $\vec{x} \in \text{span}\{\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n\}$, and thus $\mathbb{R}^n \subseteq \text{span}\{\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n\}$.

Conversely, since $\vec{e}_i \in \mathbb{R}^n$ for each $i$, $1 \leq i \leq n$ (and $\mathbb{R}^n$ is a vector space), it follows that $\text{span}\{\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n\} \subseteq \mathbb{R}^n$. The equality now follows.
Problem

Let \( \vec{u}_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}, \vec{u}_2 = \begin{bmatrix} -1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \vec{u}_3 = \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}, \vec{u}_4 = \begin{bmatrix} 1 \\ -1 \\ 1 \\ 1 \end{bmatrix} \).

Show that \( \text{span}\{ \vec{u}_1, \vec{u}_2, \vec{u}_3, \vec{u}_4 \} \neq \mathbb{R}^4 \).

Solution

If you check, you’ll find that \( \vec{e}_2 \) can not be written as a linear combination of \( \vec{u}_1, \vec{u}_2, \vec{u}_3, \) and \( \vec{u}_4 \).
Example

\[ A = \begin{bmatrix}
  0 & 1 & -1 & 2 & 5 & 1 \\
  0 & 0 & 1 & -3 & 0 & 1 \\
  0 & 0 & 0 & 0 & 1 & -2 \\
  0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix} \]

is an ref matrix.

Treat the nonzero rows of \( A \) as transposes of vectors in \( \mathbb{R}^6 \):

\[ \vec{u}_1 = \begin{bmatrix}
  0 \\
  1 \\
  -1 \\
  2 \\
  5 \\
  1
\end{bmatrix}, \quad \vec{u}_2 = \begin{bmatrix}
  0 \\
  0 \\
  1 \\
  -3 \\
  0 \\
  1
\end{bmatrix}, \quad \vec{u}_3 = \begin{bmatrix}
  0 \\
  0 \\
  0 \\
  0 \\
  1 \\
  -2
\end{bmatrix}, \]

and suppose that \( a \vec{u}_1 + b \vec{u}_2 + c \vec{u}_3 = \vec{0}_6 \) for some \( a, b, c \in \mathbb{R} \).
Example (continued)

This results in a system of six equations in three variables, whose augmented matrix is

\[
\begin{bmatrix}
0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 \\
-1 & 1 & 0 & 0 \\
2 & -3 & 0 & 0 \\
5 & 0 & 1 & 0 \\
1 & 1 & -2 & 0
\end{bmatrix}
\]

The solution to the system is easily determined to be \(a = b = c = 0\), so the set \(\{\vec{u}_1, \vec{u}_2, \vec{u}_3\}\) is independent.

In a slight abuse of terminology, we say that the nonzero rows of \(A\) are independent.

In general, the nonzero rows of any matrix in Row Echelon form (ref) form an independent set of (row) vectors.
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Theorem

Suppose $A$ is an $m \times n$ matrix with columns $\overrightarrow{a}_1, \overrightarrow{a}_2, \ldots, \overrightarrow{a}_n \in \mathbb{R}^m$. Then

1. The columns of $A$ form a linearly independent set if and only if $A\overrightarrow{x} = \overrightarrow{0}_m$ implies $\overrightarrow{x} = \overrightarrow{0}_n$.
2. The columns of $A$ span $\mathbb{R}^m$ if and only if $A\overrightarrow{x} = \overrightarrow{b}$ has a solution for every $\overrightarrow{b} \in \mathbb{R}^m$.

How is this theorem useful?

Let $\overrightarrow{x}_1, \overrightarrow{x}_2, \ldots, \overrightarrow{x}_k \in \mathbb{R}^n$.

1. Are $\overrightarrow{x}_1, \overrightarrow{x}_2, \ldots, \overrightarrow{x}_k$ linearly independent?
2. Do $\overrightarrow{x}_1, \overrightarrow{x}_2, \ldots, \overrightarrow{x}_k$ span $\mathbb{R}^n$?

To answer both questions, simply let $A$ be a matrix whose columns are the vectors $\overrightarrow{x}_1, \overrightarrow{x}_2, \ldots, \overrightarrow{x}_k \in \mathbb{R}^n$. Next, obtain the matrix $R$, which is the Row Echelon form (ref) of $A$.

- The answer to the first question is “yes” if and only if each column of $R$ has a leading one. Why?
- The answer to the second question is “yes” if and only if each row of $R$ has a leading one. Why?
Problem

Let
\[ \vec{u}_1 = [1, -1, 1, -1], \quad \vec{u}_2 = [-1, 1, 1, 1], \quad \vec{u}_3 = [1, -1, -1, 1], \quad \vec{u}_4 = [1, -1, 1, 1]. \]

Show that \( \text{span}\{\vec{u}_1, \vec{u}_2, \vec{u}_3, \vec{u}_4\} \neq \mathbb{R}^4. \)

Solution
Theorem

Let $A$ be an invertible $n \times n$ matrix. Then the columns of $A$ are independent and span $\mathbb{R}^n$. Similarly, the rows of $A$ are independent and span $\mathbb{R}^n$.

This theorem also allows us to determine if a matrix is invertible. If an $n \times n$ matrix $A$ has columns which are independent, or span $\mathbb{R}^n$, then it follows that $A$ is invertible. If it has rows that are independent, or span $\mathbb{R}^n$, then $A$ is invertible.
Problem (Again!)

Let \( \vec{u}_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix} \), \( \vec{u}_2 = \begin{bmatrix} -1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \), \( \vec{u}_3 = \begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix} \), \( \vec{u}_4 = \begin{bmatrix} 1 \\ -1 \\ 1 \\ 1 \end{bmatrix} \).

Show that \( \text{span}\{\vec{u}_1, \vec{u}_2, \vec{u}_3, \vec{u}_4\} \neq \mathbb{R}^4 \).

Solution

Let \( A = \begin{bmatrix} \vec{u}_1 & \vec{u}_2 & \vec{u}_3 & \vec{u}_4 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 1 & 1 \\ -1 & 1 & -1 & -1 \\ 1 & 1 & -1 & 1 \\ -1 & 1 & 1 & 1 \end{bmatrix} \).

The columns of \( A \) span \( \mathbb{R}^4 \) if and only if \( A \) is invertible. Since \( \det A = 0 \) (row 2 is \( -1 \) times row 1), \( A \) is not invertible, and thus \( \{\vec{u}_1, \vec{u}_2, \vec{u}_3, \vec{u}_4\} \) does not span \( \mathbb{R}^4 \).
Linear Independence

We can use the reduced row-echelon form of the matrix to determine if the columns form a linearly independent set of vectors.

Problem

*Determine whether the following set of vectors are linearly independent.*

\[ \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\} \]
Problem

Determine whether the following vectors are linearly independent. If they are linearly dependent, write one of the vectors as a linear combination of the others.

\[
\begin{bmatrix} 1 \\ 2 \\ 4 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 7 \\ 17 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 8 \\ 5 \\ 11 \\ 11 \end{bmatrix}
\]

Solution
Solution (continued)
Solution (continued)
Linear Dependence in $\mathbb{R}^n$

**Theorem**

Let $\{\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_k\}$ be a set of vectors in $\mathbb{R}^n$. Then, if $k > n$ then the set is linearly dependent.
Subspaces

Theorem (Subspace Test)

A subset $V$ of $\mathbb{R}^n$ is a subspace of $\mathbb{R}^n$ if

1. the zero vector of $\mathbb{R}^n$, $\vec{0}_n$, is in $V$;
2. $V$ is closed under addition, i.e., for all $\vec{u}, \vec{w} \in V$, $\vec{u} + \vec{w} \in V$;
3. $V$ is closed under scalar multiplication, i.e., for all $\vec{u} \in V$ and $k \in \mathbb{R}$, $k \vec{u} \in V$.

The subset $V = \{ \vec{0}_n \}$ is a subspace of $\mathbb{R}^n$ (verify this), as is the set $\mathbb{R}^n$ itself. Any other subspace of $\mathbb{R}^n$ is a proper subspace of $\mathbb{R}^n$.

Notation

If $V$ is a subset of $\mathbb{R}^n$, we write $V \subseteq \mathbb{R}^n$. In some texts, for a saying that $V$ is a subspace of $\mathbb{R}^n$, the notation used is $V \leq \mathbb{R}^n$. 

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Problem

Is $V = \left\{ \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \bigg| \begin{array}{c} a, b, c, d \in \mathbb{R} \text{ and } 2a - b = c + 2d \end{array} \right\}$ a subspace of $\mathbb{R}^4$? Justify your answer.

Solution
Solution (continued)
Solution (continued)
Subspaces

Definition

Let $V$ be a nonempty collection of vectors in $\mathbb{R}^n$. Then $V$ is a subspace if whenever $a$ and $b$ are scalars and $\mathbf{u}$ and $\mathbf{v}$ are vectors in $V$, $a\mathbf{u} + b\mathbf{v}$ is also in $V$.

Subspaces are closely related to the span of a set of vectors which we discussed earlier.

Theorem

Let $V$ be a nonempty collection of vectors in $\mathbb{R}^n$. Then $V$ is a subspace of $\mathbb{R}^n$ if and only if there exist vectors $\{\mathbf{u}_1, ..., \mathbf{u}_k\}$ in $V$ such that

$$V = \text{span} \{\mathbf{u}_1, ..., \mathbf{u}_k\}$$
Subspaces

Subspaces are also related to the property of linear independence.

**Theorem**

If $V$ is a subspace of $\mathbb{R}^n$, then there exist linearly independent vectors $\{\vec{u}_1, ..., \vec{u}_k\}$ of $V$ such that

$$V = \text{span}\{\vec{u}_1, ..., \vec{u}_k\}$$

In other words, subspaces of $\mathbb{R}^n$ consist of spans of finite, linearly independent collections of vectors in $\mathbb{R}^n$. 
Problem

Is \( V = \left\{ \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \mid a, b, c, d \in \mathbb{R} \text{ and } 2a - b = c + 2d \right\} \) a subspace of \( \mathbb{R}^4 \)? Justify your answer.

Solution 2
Basis of a Subspace

**Definition**

Let $V$ be a subspace of $\mathbb{R}^n$. Then $\{\vec{u}_1, ..., \vec{u}_k\}$ is called a **basis** for $V$ if the following conditions hold:

- $\text{span}\{\vec{u}_1, ..., \vec{u}_k\} = V$
- $\{\vec{u}_1, ..., \vec{u}_k\}$ is linearly independent.
Example

The subset \( \{ \vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n \} \) is a basis of \( \mathbb{R}^n \), called the standard basis of \( \mathbb{R}^n \). (We’ve already seen that \( \{ \vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n \} \) is linearly independent and that \( \mathbb{R}^n = \text{span}\{ \vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n \} \).

Example

In a previous problem, we saw that \( \mathbb{R}^4 = \text{span}(S) \) where

\[
S = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}.
\]

\( S \) is also linearly independent (prove this). Therefore, \( S \) is a basis of \( \mathbb{R}^4 \).
The following theorem claims that any two bases of a subspace must be of the same size.

**Theorem**

Let $V$ be a subspace of $\mathbb{R}^n$ and suppose $\{\vec{u}_1, \ldots, \vec{u}_k\}$ and $\{\vec{v}_1, \ldots, \vec{v}_m\}$ are two bases for $V$. Then $k = m$.

The previous theorem shows than all bases of a subspace will have the same size. This size is called the **dimension** of the subspace.

**Definition**

Let $V$ be a subspace of $\mathbb{R}^n$. Then the **dimension** of $V$ is the number of a vectors in a basis of $V$. 
Properties of $\mathbb{R}^n$

Note that the dimension of $\mathbb{R}^n$ is $n$.

There are some other important properties of vectors in $\mathbb{R}^n$.

**Theorem**

- If $\{\vec{u}_1, \ldots, \vec{u}_n\}$ is a linearly independent set of vectors in $\mathbb{R}^n$, then $\{\vec{u}_1, \ldots, \vec{u}_n\}$ is a basis for $\mathbb{R}^n$.
- Suppose $\{\vec{u}_1, \ldots, \vec{u}_m\}$ spans $\mathbb{R}^n$. Then $m \geq n$.
- If $\{\vec{u}_1, \ldots, \vec{u}_n\}$ spans $\mathbb{R}^n$, then $\{\vec{u}_1, \ldots, \vec{u}_n\}$ is linearly independent.
Problem

Let

\[ U = \left\{ \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \in \mathbb{R}^4 \mid a - b = d - c \right\}. \]

Show that \( U \) is a subspace of \( \mathbb{R}^4 \), find a basis of \( U \), and find \( \text{dim}(U) \).
Solution
Solution (continued)
Theorem

The following properties hold in $\mathbb{R}^n$:

1. Suppose $\{\vec{u}_1, \cdots, \vec{u}_n\}$ is linearly independent. Then $\{\vec{u}_1, \cdots, \vec{u}_n\}$ is a basis for $\mathbb{R}^n$.
2. Suppose $\{\vec{u}_1, \cdots, \vec{u}_m\}$ spans $\mathbb{R}^n$. Then $m \geq n$.
3. If $\{\vec{u}_1, \cdots, \vec{u}_n\}$ spans $\mathbb{R}^n$, then $\{\vec{u}_1, \cdots, \vec{u}_n\}$ is linearly independent.

Question

What is the significance of this result?
Answer

Let $V$ be a subspace of $\mathbb{R}^n$ and suppose $B \subseteq V$.

- If $B$ spans $V$ and $|B| = \dim(V)$, then $B$ is also independent, and hence $B$ is a basis of $V$.

- If $B$ is independent and $|B| = \dim(V)$, then $B$ also spans $V$, and hence $B$ is a basis of $V$.

Therefore if $|B| = \dim(V)$, it is sufficient to prove that $B$ is either independent or spans $V$ in order to prove it is a basis.
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Row and Column Space

Definition

Let $A$ be an $m \times n$ matrix. The column space of $A$ is the span of the columns of $A$. The row space of $A$ is the span of the rows of $A$.

Problem

Find the rank of the matrix $A$ and describe the column and row spaces efficiently.

$$A = \begin{bmatrix} 1 & 2 & 1 & 3 & 2 \\ 1 & 3 & 6 & 0 & 2 \\ 3 & 7 & 8 & 6 & 6 \end{bmatrix}$$
Example: Column Space

Solution
Example: Column Space

Solution (continued)
Example: Row Space

Solution (continued)

Notice that the vectors used in the description of the column space are from the original matrix, while those in the row space are from the reduced row-echelon form or ALSO can be from the original matrix.
Null Space

Definition

Let $A$ be an $m \times n$ matrix. The null space of $A$, or kernel of $A$ is defined as:

$$\text{ker}(A) = \{ X \in \mathbb{R}^n : AX = 0 \}$$

Theorem

Let $A$ be and $m \times n$ matrix. The kernel of $A$, $\text{ker}(A)$, is a subspace of $\mathbb{R}^n$.

Definition

The dimension of the null space of a matrix is called the nullity, denoted $\text{null}(A)$.

To find $\text{ker}(A)$, we solve the system of equations $AX = 0$. 
Problem

Find ker(A) for the matrix A:

\[ A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & 1 \\ 2 & 3 & 3 \end{bmatrix} \]

Solution
Null Space

Solution (continued)
Image of a Matrix

Definition

The image of an $m \times n$ matrix $A$ is defined as:

$$\text{im}(A) = \{ Y \in \mathbb{R}^m : \text{there exists an } X \in \mathbb{R}^n \text{ such that } AX = Y \}$$

Roughly, the image of $A$ are the vectors of $\mathbb{R}^m$ which “get hit” by $A$.

Theorem

Let $A$ be an $m \times n$ matrix. The image of $A$, $\text{im}(A)$, is a subspace of $\mathbb{R}^m$.

It can be shown that $\text{im}(A) = \text{col}(A)$. Thus, to find $\text{im}(A)$, we just find the column space of $A$, that is, $\text{col}(A)$.
Problem

Find $\text{im}(A)$ for the matrix $A$:

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & 1 \\ 2 & 3 & 3 \end{bmatrix}$$

Solution

As we saw before, the reduced row-echelon form is:

$$\begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$
The Rank-Nullity Theorem

One of the most important theorems in Linear Algebra with tremendous consequences is the following:

**Theorem**

Let $A$ be an $m \times n$ matrix. Then,

$$\text{rank}(A) + \text{null}(A) = n$$

For instance, in the last example, $A$ was a $3 \times 3$ matrix. The rank was 2 (since the image or column space had dimension 2) and the nullity was 1 (since the null space had dimension 1). Then,

$$\text{rank}(A) + \text{null}(A) = 2 + 1 = 3 = n$$
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Definitions (Recall)

Let \( \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \) and \( \mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} \) be vectors in \( \mathbb{R}^n \).

1. The dot product of \( \mathbf{x} \) and \( \mathbf{y} \) is

\[
\mathbf{x} \cdot \mathbf{y} = \mathbf{x}^T \mathbf{y} = x_1y_1 + x_2y_2 + \cdots + x_ny_n = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}^T \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}
\]

Note: \( \mathbf{x} \cdot \mathbf{y} \) is a \( 1 \times 1 \) matrix, but we also treat it as a scalar.

2. The length of \( \mathbf{x} \), denoted \( ||\mathbf{x}|| \) is

\[
||\mathbf{x}|| = \sqrt{\mathbf{x}^T \mathbf{x}} = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2} = \sqrt{\mathbf{x} \cdot \mathbf{x}}
\]

3. \( \mathbf{x} \) is called a unit vector if \( ||\mathbf{x}|| = 1 \).
Orthogonality

Definitions

1. Let $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$. We say the $\mathbf{u}$ and $\mathbf{v}$ are orthogonal if $\mathbf{u} \cdot \mathbf{v} = 0$.

2. Let $\{\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_m\}$ be a set of vectors in $\mathbb{R}^n$. Then this set is called an orthogonal set if:
   1. $\mathbf{u}_i \cdot \mathbf{u}_j = 0$ for all $i \neq j$
   2. $\mathbf{u}_i \neq 0$ for all $i$

3. A set of vectors, $\{\mathbf{w}_1, \cdots, \mathbf{w}_m\}$ is said to be an orthonormal set if

\[
\mathbf{w}_i \cdot \mathbf{w}_j = \delta_{ij} = \begin{cases} 
1 & \text{if } i = j \\
0 & \text{if } i \neq j 
\end{cases}
\]
Theorem (Properties of length and the dot product)

Let $k$ and $p$ denote scalars and $\vec{u}, \vec{v}, \vec{w}$ denote vectors. Then the dot product $\vec{u} \cdot \vec{v}$ satisfies the following properties.

- $\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$
- $\vec{u} \cdot \vec{u} \geq 0$ and equals zero if and only if $\vec{u} = \vec{0}$
- $(k\vec{u} + p\vec{v}) \cdot \vec{w} = k(\vec{u} \cdot \vec{w}) + p(\vec{v} \cdot \vec{w})$
- $\vec{u} \cdot (k\vec{v} + p\vec{w}) = k(\vec{u} \cdot \vec{v}) + p(\vec{u} \cdot \vec{w})$
- $||\vec{u}||^2 = \vec{u} \cdot \vec{u}$
Example

Let \( \{ \vec{x}_1, \vec{x}_2, \ldots, \vec{x}_k \} \in \mathbb{R}^n \) and suppose \( \mathbb{R}^n = \text{span}\{ \vec{x}_1, \vec{x}_2, \ldots, \vec{x}_k \} \). Furthermore, suppose that there exists a vector \( \vec{u} \in \mathbb{R}^n \) for which \( \vec{u} \cdot \vec{x}_j = 0 \) for all \( j, 1 \leq j \leq k \). What type of vector is \( \vec{u} \)?

Solution

Write \( \vec{u} = t_1 \vec{x}_1 + t_2 \vec{x}_2 + \cdots + t_k \vec{x}_k \) for some \( t_1, t_2, \ldots, t_k \in \mathbb{R} \) (this is possible because \( \vec{x}_1, \vec{x}_2, \ldots, \vec{x}_k \) span \( \mathbb{R}^n \)).

Then

\[
\| \vec{u} \|^2 = \vec{u} \cdot \vec{u} \\
= \vec{u} \cdot (t_1 \vec{x}_1 + t_2 \vec{x}_2 + \cdots + t_k \vec{x}_k) \\
= \vec{u} \cdot t_1 \vec{x}_1 + \vec{u} \cdot t_2 \vec{x}_2 + \cdots + \vec{u} \cdot t_k \vec{x}_k \\
= t_1 (\vec{u} \cdot \vec{x}_1) + t_2 (\vec{u} \cdot \vec{x}_2) + \cdots + t_k (\vec{u} \cdot \vec{x}_k) \\
= t_1(0) + t_2(0) + \cdots + t_k(0) = 0
\]

Since \( \| \vec{u} \|^2 = 0 \), \( \| \vec{u} \| = 0 \). We know that \( \| \vec{u} \| = 0 \) if and only if \( \vec{u} = \vec{0}_n \). Therefore, \( \vec{u} = \vec{0}_n \).
Examples

1. The standard basis of $\mathbb{R}^n$ is an orthonormal set (and hence an orthogonal set).

2. 
   \[
   \left\{\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix}\right\}
   \]
   is an orthogonal (but not orthonormal) subset of $\mathbb{R}^4$.

3. 
   \[
   \left\{\frac{1}{2} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \frac{1}{2} \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}, \frac{1}{2} \begin{bmatrix} 1 \\ -1 \end{bmatrix}\right\}
   \]
   is an orthonormal subset of $\mathbb{R}^4$. 
Definition

Normalizing an orthogonal set is the process of turning an orthogonal (but not orthonormal) set into an orthonormal set. If \( \{ \vec{u}_1, \vec{u}_2, \ldots, \vec{u}_k \} \) is an orthogonal subset of \( \mathbb{R}^n \), then

\[
\left\{ \frac{1}{||\vec{u}_1||} \vec{u}_1, \frac{1}{||\vec{u}_2||} \vec{u}_2, \ldots, \frac{1}{||\vec{u}_k||} \vec{u}_k \right\}
\]

is an orthonormal set.

Problem

Consider the set of vectors given by

\[
\{ \vec{u}_1, \vec{u}_2 \} = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}
\]

Show that it is an orthogonal set of vectors but not an orthonormal one. Find the corresponding orthonormal set.
Solution
Orthogonal Matrix

**Definition**

A real $n \times n$ matrix $U$ is called an **orthogonal matrix** if

$$UU^T = U^T U = I$$

**Problem**

Show the matrix $U$ is orthogonal.

$$U = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$$
A matrix is **orthogonal** if its rows (or columns) form an **orthonormal** set of vectors.

**Theorem (Orthonormal Basis)**

The rows of an $n \times n$ orthogonal matrix form an orthonormal basis of $\mathbb{R}^n$. Further, any orthonormal basis of $\mathbb{R}^n$ can be used to construct an $n \times n$ orthogonal matrix.

**Theorem (Determinant of Orthogonal Matrices)**

Suppose $U$ is an orthogonal matrix. Then $\det(U) = \pm 1$. 
Theorem

Let \( \{\vec{w}_1, \vec{w}_2, \cdots, \vec{w}_k\} \) be an orthonormal set of vectors in \( \mathbb{R}^n \). Then, this set is linearly independent and forms a basis for the subspace \( \mathcal{W} = \text{span}\{\vec{w}_1, \vec{w}_2, \cdots, \vec{w}_k\} \).

Proof.

To show it is a linearly independent set, suppose a linear combination of these vectors equals \( \vec{0} \), such as:

\[
a_1 \vec{w}_1 + a_2 \vec{w}_2 + \cdots + a_k \vec{w}_k = \vec{0}, \quad a_i \in \mathbb{R}
\]

We need to show that all \( a_i = 0 \). To do so, take the dot product of each side of the above equation with the vector \( \vec{w}_i \) and obtain the following.

\[
\vec{w}_i \cdot (a_1 \vec{w}_1 + a_2 \vec{w}_2 + \cdots + a_k \vec{w}_k) = \vec{w}_i \cdot \vec{0}
\]

\[
a_1(\vec{w}_i \cdot \vec{w}_1) + a_2(\vec{w}_i \cdot \vec{w}_2) + \cdots + a_k(\vec{w}_i \cdot \vec{w}_k) = 0
\]
Continued.

Now since the set is orthogonal, $\vec{w}_i \cdot \vec{w}_m = 0$ for all $m \neq i$, so we have:

$$a_1(0) + \cdots + a_i(\vec{w}_i \cdot \vec{w}_i) + \cdots + a_k(0) = 0$$

$$a_i ||\vec{w}_i||^2 = 0$$

Since the set is orthogonal, we know that $||\vec{w}_i||^2 \neq 0$. It follows that $a_i = 0$. Since the $a_i$ was chosen arbitrarily, the set $\{\vec{w}_1, \vec{w}_2, \cdots, \vec{w}_k\}$ is linearly independent.

Finally since $W = \text{span}\{\vec{w}_1, \vec{w}_2, \cdots, \vec{w}_k\}$, the set of vectors also spans $W$ and therefore forms a basis of $W$. □
We have already seen that an orthonormal set of vectors in $\mathbb{R}^n$ is a linearly independent set. However, we are interested in the opposite problem.

**Question:** If given a linearly independent set of vectors $\mathbf{u}_1, \cdots, \mathbf{u}_k \in \mathbb{R}^n$, how do we construct a corresponding orthogonal set? Corresponding orthonormal set?

**Answer:** The Gram-Schmidt Process
Let \{\vec{u}_1, ..., \vec{u}_k\} be a set of linearly independent vectors in \(\mathbb{R}^n\).

I. Construct a new set of vectors \{\vec{v}_1, ..., \vec{v}_k\} as follows:

\[
\begin{align*}
\vec{v}_1 &= \vec{u}_1 \\
\vec{v}_2 &= \vec{u}_2 - \left( \frac{\vec{u}_2 \cdot \vec{v}_1}{\| \vec{v}_1 \|^2} \right) \vec{v}_1 \\
\vec{v}_3 &= \vec{u}_3 - \left( \frac{\vec{u}_3 \cdot \vec{v}_1}{\| \vec{v}_1 \|^2} \right) \vec{v}_1 - \left( \frac{\vec{u}_3 \cdot \vec{v}_2}{\| \vec{v}_2 \|^2} \right) \vec{v}_2 \\
&\quad \vdots \\
\vec{v}_k &= \vec{u}_k - \left( \frac{\vec{u}_k \cdot \vec{v}_1}{\| \vec{v}_1 \|^2} \right) \vec{v}_1 - \left( \frac{\vec{u}_k \cdot \vec{v}_2}{\| \vec{v}_2 \|^2} \right) \vec{v}_2 - \cdots - \left( \frac{\vec{u}_k \cdot \vec{v}_{k-1}}{\| \vec{v}_{k-1} \|^2} \right) \vec{v}_{k-1}
\end{align*}
\]

Then \{\vec{v}_1, ..., \vec{v}_k\} is an orthogonal set.
II. Now, let $\mathbf{w}_i = \frac{\mathbf{v}_i}{||\mathbf{v}_i||}$ for $i = 1, \ldots, k$. Then, $\{\mathbf{w}_1, \ldots, \mathbf{w}_k\}$ is an orthonormal set.

The result of the Gram-Schmidt Process is three related sets, where:

- $\{\mathbf{u}_1, \ldots, \mathbf{u}_k\}$ is the original set
- $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is the corresponding orthogonal set
- $\{\mathbf{w}_1, \ldots, \mathbf{w}_k\}$ is the corresponding orthonormal set.

Notice that $\text{span}\{\mathbf{u}_1, \ldots, \mathbf{u}_k\} = \text{span}\{\mathbf{v}_1, \ldots, \mathbf{v}_k\} = \text{span}\{\mathbf{w}_1, \ldots, \mathbf{w}_k\}$.

Therefore, we can use the Gram-Schmidt Process to find orthogonal or orthonormal sets which have the same span as the original linearly independent set.
Problem

Let \( \{ \vec{u}_1, \vec{u}_2 \} = \{ [1, 1, 0]^T, [3, 2, 0]^T \} \). Find an orthonormal set of vectors \( \{ \vec{w}_1, \vec{w}_2 \} \) having the same span.

Solution
Orthonormal Set

Solution (continued)