Mycielski's Theorem

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Introduction

The goal here is to present Mycielski's theorem and a proof. First, Mycielski's construction will be presented. Then the theorem will be stated and a theorem will be given. Examples on special cases of Mycielski's construction and theorem will be presented for illustration purposes.

Mycielski's construction

This construction is defined on simple graphs. Let $G$ be such a graph with $V(G) = \{v_1, \ldots, v_n\}$. Let $W = \{u_1, \ldots, u_n\} \cup \{w\}$. A graph, $G'$, gotten by this construction, by adding the vertices of $W$ to $G$ such that $u_i$ is adjacent to all of the vertices of $N_G(v_i)$, and $w$ is adjacent to all of the vertices of $U = \{u_1, \ldots, u_n\}$.

Example: The yield of a 2-chromatic graph $K_2$.

The Statement of The Theorem (Mycielski [1955])

Starting with a $k$-chromatic triangle-free graph $G$, constructing a graph, using Mycielski's construction, produces a $k + 1$-chromatic triangle-free graph $G'$.

Example: The yield of a 3-chromatic graph $C_5$.
**Ugly Proof:**

Assume that $G'$ has a triangle, containing the vertex $u_i$. This vertex is adjacent to the vertices that the $v_i$ are adjacent to, but this would imply that $G$ has a triangle, where the vertices that $u_i$ forms a triangle with, are vertices of this triangle along with $v_i$. This would contradict that $G$ is triangle-free.

A proper $k$-coloring $f$ of $G$ extends to a proper $k + 1$-coloring of $G'$ by setting $f(u_i) = f(v_i)$ and $f(w) = k + 1$; hence $\chi(G') \leq \chi(G) + 1$. We prove equality by showing that $\chi(G) < \chi(G')$. To prove this consider a proper $k + 1$-coloring of $G'$ on and obtain from it a proper $k$-coloring of $G$.

Let $g$ be a proper $k + 1$-coloring of $G'$. By changing the names of colors, we may assume that $g(w) = k + 1$. This restricts $g$ to $\{1, ..., k\}$ on $U$. On $V(G)$, all $k + 1$ colors may be used. Let $A$ be a set of vertices in $G$ on which $g$ uses color $k + 1$; we change the colors used on $A$ to obtain a proper $k$-coloring of $G$.

For each $v_i \in A$, it is shown that a change in color of $v_i$ to $g(u_i)$ is possible. So we show that with this new color, vertices adjacent to $v_i \in A$ have a color different from $g(u_i)$. First notice that because all vertices of $A$ have the same color, no two vertices of $A$ are adjacent. Second, since edges $v' \in V(G) - A$ adjacent to $v_i \in A$ are also adjacent to $u_i$, where $u_i$ is not adjacent to $v_i \in A$ by the construction, and since $g(u_i) \neq g(v')$, by construction, the new color of $v_i \in A$ is not the color $g(v')$. We have shown that the modified coloring of $V(G)$ is a proper $k$-coloring of $G$.

**Reference**

West, D. “Introduction to Graph Theory, Second Edition.” pp.205-206