Hereditary Families and the Dominance Order

Michael D. Barrus

Department of Mathematics
University of Rhode Island

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Partitions and degree sequences

6, 5 + 1, 4 + 2, 4 + 1 + 1, 3 + 3, 3 + 2 + 1, 3 + 1 + 1 + 1
2 + 2 + 2, 2 + 2 + 1 + 1, 2 + 1 + 1 + 1 + 1, 1 + 1 + 1 + 1 + 1 + 1

\( d(G) = (4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 2, 2, 2, 2, 2, 2, 2, 2, 2) \)

also written \((4, 22, 3, 8, 2, 6)\) — a graphic partition of 124
Partitions and degree sequences

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also written \((4^{22}, 3^8, 2^6)\) — a graphic partition of 124
The dominance order

Majorization:
Given \( d = (d_1, \ldots, d_n) \) and \( e = (e_1, \ldots, e_m) \),
we say \( d \) majorizes \( e \), written \( d \succcurlyeq e \),
if \( \sum d_i = \sum e_i \) and \( \sum_{i=1}^{k} d_i \geq \sum_{i=1}^{k} e_i \) for all \( k \).

Example: \((3, 2, 1) \succeq (2, 2, 1, 1)\),
since
\[
3 > 2, \quad 3 + 2 > 2 + 2,
3 + 2 + 1 > 2 + 2 + 1,
3 + 2 + 1 + 0 = 2 + 2 + 1 + 1.
\]
The dominance order

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Graphic partitions form an ideal (down-set) in the partitions of \( 2m \).
If $d \succeq e$, then $e$ has at least as many labeled realizations as $d$.

$(5, 1^5)$: 1 realization
$(3^2, 1^4)$: 6 realizations
$(2^3, 1^4)$: 75 realizations
$(1^{10})$: 945 realizations
Life’s different(?) at the top
Hamiltonicity of realization graphs [Arikati, 1999]

\[ d = (2, 2, 2, 1, 1) \]

\[
\begin{align*}
V(R(d)) &= \{ \text{realizations of } d \}, \\
E(R(d)) &= \{ \text{pairs joined by a 2-switch} \}
\end{align*}
\]
Life’s different(?) at the top
Hamiltonicity of realization graphs \cite{Arikati, 1999}

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\[ d = (2, 2, 2, 1, 1) \]

If \( d \) is \( \leq 1 \) step down in the dominance order on graphic sequences, then the realization graph of \( d \) is Hamiltonian.

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Graph classes at the top

Degree sequences for the following classes are “upwards closed” in the dominance order:

- **Threshold graphs** [Ruch–Gutman, 1979; Peled–Srinivasan, 1989]
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- **Split graphs** [Merris, 2003]
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Threshold and split graphs: forbidden subgraphs

The classes of threshold and split graphs are **hereditary** — closed under taking induced subgraphs.
Threshold and split graphs: forbidden subgraphs

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Equivalently, they have forbidden induced subgraph characterizations...

- **Threshold graphs**: \(\{2K_2, P_4, C_4\}\)-free

- **Split graphs**: \(\{2K_2, C_4, C_5\}\)-free
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The classes of threshold and split graphs are **hereditary** — closed under taking induced subgraphs.

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Is it the **forbidden subgraphs** putting the degree sequences at the top of the dominance order? Other examples?
Erdős–Gallai conditions and differences

[Erdős–Gallai, 1960]

A list \((d_1, \ldots, d_n)\) of nonnegative integers in descending order with even sum is a degree sequence if and only if

\[
\sum_{i \leq k} d_i \leq k(k - 1) + \sum_{i > k} \min\{k, d_i\}
\]

for all \(k\).
Erdős–Gallai conditions and differences


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for all \(k \leq \max\{i : d_i \geq i - 1\}\).
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**Erdős–Gallai differences:**

\[
\Delta_k(d) = k(k - 1) + \sum_{i > k} \min\{k, d_i\} - \sum_{i \leq k} d_i
\]

\[
\text{RHS}_k(d) = k(k - 1) + \sum_{i > k} \min\{k, d_i\}
\]

\[
\text{LHS}_k(d) = \sum_{i \leq k} d_i
\]
Threshold and split graphs: degree sequences

\[ \Delta_k(d) = k(k - 1) + \sum_{i > k} \min\{k, d_i\} - \sum_{i \leq k} d_i \]

- **Threshold graphs:** [Hammer–Ibaraki–Simeone, 1978]
  \[ \Delta_k(d) = 0 \text{ for } 1 \leq k \leq \max\{i : d_i \geq i - 1\} \]

- **Split graphs:** [Tyshkevich, 1980; Hammer–Simeone, 1981]
  \[ \Delta_m(d) = 0 \text{ for } m = \max\{i : d_i \geq i - 1\} \]
A connection to dominance orders

Lemma (B, 2018+)

If \( d \succeq e \), then the Erdős–Gallai differences of \( d \) can be matched up with those of \( e \) such that \( \Delta_i(d) \leq \Delta_{i'}(e) \).

\[(4,2,2,1,1):\]
\[0, 0, 0, 4, 10\]

\[(3,3,1,1,1,1):\]
\[2, 0, 2, 6, 12, 20\]

\[(1,1,1,1,1,1,1,1,1,1):\]
\[8, 8, 10, 14, 20, 28, 38, 50, 64, 80\]
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(1,1,1,1,1,1,1,1,1,1):
8, 8, 10, 14, 20, 28, 38, 50, 64, 80
For $d = (3, 3, 1, 1, 1, 1)$: $\{2, 0, 2, 6, 12, 20\}$

$\Delta_k(d) = 0$ iff the structure

![Diagram of graph with vertices labeled 1 through k (clique) and one vertex labeled k (independent set), connected by an edge to the rest of the graph.](image)
EG-differences of 0

For \( d = (3, 3, 1, 1, 1, 1) \): 2, 0, 2, 6, 12, 20

\[ \Delta_k(d) = 0 \text{ iff the structure} \]

![Diagram](Image)
EG-differences of 0


For \( d = (3, 3, 1, 1, 1, 1) \):

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Call a graph or sequence **indecomposable** if no such partition occurs with a nonempty “Rest of the graph”.
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Decomposable sequences form an “upward closed” portion of the dominance order.
EG-differences of 0

For \( d = (9, 9, 7, 7, 6, 6, 6, 3, 3, 1, 1) \),

\[
\Delta_2(d) = 2 \cdot 1 + 2 + 2 + 2 + 2 + 2 + 2 + 2 + 1 + 1 - (9 + 9) = 0
\]

\[
\Delta_4(d) = 4 \cdot 3 + 4 + 4 + 4 + 3 + 3 + 1 + 1 - (9 + 9 + 7 + 7) = 0
\]

\[
(9, 9, 7, 7, 6, 6, 6, 3, 3, 1, 1) = (2, 2; 1, 1) \circ (2, 2; 1, 1) \circ (2, 2, 2)
\]
If $d \succeq e$, then the Erdős–Gallai differences of $d$ can be matched up with those of $e$ such that $\Delta_i(d) \leq \Delta_i(e)$. 

**Lemma (B, 2018+)**
Lemma (B, 2018+) If $d \succeq e$, then the Erdős–Gallai differences of $d$ can be matched up with those of $e$ such that $\Delta_i(d) \leq \Delta_i(e)$.

But what about other hereditary families of graphs?
An experiment: weakly threshold graphs

Define a **weakly threshold sequence** to be list \( d = (d_1, \ldots, d_n) \) of nonnegative integers in descending order having even sum and satisfying

\[
\Delta_k(d) = RHS_k(d) - LHS_k(d) \leq 1
\]

for all \( k \leq \max\{i : d_i \geq i - 1\} \).
An experiment: weakly threshold graphs

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for all \( k \leq \max\{i : d_i \geq i - 1\} \).

A **weakly threshold graph** will be a graph having a weakly threshold sequence as its degree sequence.
Forbidden induced subgraphs?

Theorem (B, 2018)
A graph $G$ is weakly threshold if and only if it is

\[
\{2K_2, C_4, C_5, H, H, S_3, S_3\}
\]

-free. Weakly threshold graphs form a large subclass of interval $\cap$ co-interval. (The latter class's forbidden induced subgraphs:)

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Forbidden induced subgraphs?

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A graph $G$ is weakly threshold if and only if it is \{2$K_2$, $C_4$, $C_5$, $H$, $\overline{H}$, $S_3$, $\overline{S_3}$\} -free.

Weakly threshold graphs form a large subclass of $\text{interval} \cap \text{co-interval}$. (The latter class's forbidden induced subgraphs:)

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Forbidden induced subgraphs

- Threshold sequences / graphs
- Weakly threshold sequences / graphs
- More generally,
Forbidden induced subgraphs

- Threshold sequences / graphs

- Weakly threshold sequences / graphs

- More generally, Erdős–Gallai differences before and after vertex deletions behave nicely... so suitable “EG-difference profiles” do correspond to certain forbidden induced subgraph sets!
Iterative construction results

- Threshold sequences / graphs
- Weakly threshold sequences / graphs?
- More generally?
Iterative construction: threshold graphs?

$G$ is a threshold graph if and only if $G$ can be constructed from a single vertex via these operations.

- Adding a dominating vertex
- Adding an isolated vertex

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Iterative construction: threshold graphs?

Options for adding:
- Dominating vertex
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Iterative construction: weakly threshold graphs

Theorem (B, 2018)

G is a weakly threshold graph if and only if G can be constructed by beginning with a single vertex and iteratively adding:

- a dominating vertex, or
- an isolated vertex
Iterative construction: weakly threshold graphs

Theorem (B, 2018)

$G$ is a **weakly** threshold graph if and only if $G$ can be constructed by beginning with a single vertex or $P_4$ and iteratively adding

- a dominating vertex, or
- an isolated vertex, or
- a **weakly dominating** vertex, or
- a weakly isolated vertex, or
- a semi-joined $P_4$.  

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Hereditary Families and the Dominance Order
Iterative construction results

- **Threshold sequences / graphs:** (Constructed from • via dominating/isolated vertices.)

- **Weakly threshold sequences / graphs:**
  (Constructed from a single vertex or $P_4$ via (weakly?) dominating/isolated vertices and/or semijoined $P_4$’s)

- More generally,
Iterative construction results

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- **More generally,** we can use the composition operation and forbidden subgraphs to come up with iterative construction algorithms for given “EG-difference profiles.”
Enumeration of sequences and graphs

- **Threshold sequences / graphs:** $\frac{1}{2} \cdot 2^n$
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- **More generally?**
Enumeration: sequences

\[ a_n = \text{number of weakly threshold sequences of length } n \]

\( (1, 1, 2, 4, 9, 21, 50, 120, 289, 697, 1682, 4060, \ldots) \)

It's in OEIS.org! Sequences A024537, A171842

Binomial transform of \( 1, 0, 1, 0, 2, 0, 4, 0, 8, 0, 16, \ldots \)

Number of nonisomorphic \( n \)-element interval orders with no 3-element antichain.

Top left entry of the \( n \)th power of

\[
\begin{pmatrix}
1 & 1 & 0 \\
1 & 1 & 1 \\
0 & 1 & 1 \\
\end{pmatrix}
\]

or of

\[
\begin{pmatrix}
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1 & 1 & 1 \\
\end{pmatrix}
\]

Number of \((1, s_1, \ldots, s_{n-1}, 1)\) such that \( s_i \in \{1, 2, 3\} \) and \(|s_i - s_{i-1}| \leq 1\).

Partial sums of the Pell numbers prefaced with a 1.

The number of ways to write an \((n-1)\)-bit binary sequence and then give runs of ones weakly incrementing labels starting with 1, e.g., 0011010011022203003330044040055555.

Lower bound of the order of the set of equivalent resistances of \((n-1)\) equal resistors combined in series and in parallel.

Proposition: For all \( n \geq 4 \),

\[ a_n = 4a_{n-1} - 4a_{n-2} + a_{n-4} \]

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- Number of \((1, s_1, \ldots, s_{n-1}, 1)\) such that \(s_i \in \{1, 2, 3\}\) and \(|s_i - s_{i-1}| \leq 1\).
- Partial sums of the Pell numbers prefaced with a 1.
- The number of ways to write an \((n - 1)\)-bit binary sequence and then give runs of ones weakly incrementing labels starting with 1, e.g., 001101001102220300333044040055555.
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**Proposition:** For all \( n \geq 4 \), \( a_n = 4a_{n-1} - 4a_{n-2} + a_{n-4} \).
Enumeration: graphs

\[ b_n = \text{number of weakly threshold graphs with } n \text{ vertices} \]

\[ \text{# WT graphs with exactly } k \text{ indecomposable parts: } H(x)^{k-1}(H(x) - x) \]

where \( H(x) = \sum_{n=1}^{\infty} h_n x^n \) and \( h_n = \text{# indecomposable WT on } n \text{ vertices} \)

(which satisfies \( h_n = 3h_{n-1} - h_{n-2} \))
Enumeration: graphs

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**Theorem**

*The generating function for \((b_n)\) is given by*

\[
\sum_{n=0}^{\infty} b_n x^n = \frac{x - 2x^2 - x^3 - x^5}{1 - 4x + 3x^2 + x^3 + x^5}.
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\]

\[ b_n = c_1 \left( \frac{1 + \sqrt{5}}{2} \right)^n + c_2 \left( \frac{1 - \sqrt{5}}{2} \right)^n 
+ c_3 \left( \frac{6 - (1 + i\sqrt{3})(27 - 3\sqrt{57})^{1/3} - (1 - i\sqrt{3})(27 + 3\sqrt{57})^{1/3}}{6} \right)^n 
+ c_4 \left( \frac{6 - (1 - i\sqrt{3})(27 - 3\sqrt{57})^{1/3} - (1 + i\sqrt{3})(27 + 3\sqrt{57})^{1/3}}{6} \right)^n 
+ c_5 \left( \frac{3 + (27 - 3\sqrt{57})^{1/3} + (27 + 3\sqrt{57})^{1/3}}{3} \right)^n, \]
Enumeration of sequences and graphs

- **Threshold sequences / graphs:** \( \frac{1}{2} \cdot 2^n \)
  (Constructed from \( \bullet \) via dominating/isolated vertices.)

- **Weakly threshold sequences / graphs:**
  
  \[ a_n \sim \frac{1}{4} (1 + \sqrt{2})^n \geq \frac{1}{4} \cdot 2.4^n \]

  \[ b_n \sim c_5 \left( \frac{3 + (27 - 3\sqrt{57})^{1/3} + (27 + 3\sqrt{57})^{1/3}}{3} \right)^n \approx 0.096 \cdot 2.7^n \]
  
  (Constructed from a single vertex or \( P_4 \) by iteratively adding one of ...)

- **More generally...**
Enumeration of sequences and graphs

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  \]
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- **More generally...**
In summary

Some of the best properties of threshold and split graphs are shared by other hereditary families whose degree sequences form upward-closed subsets of the dominance order.

What other properties?

What other approaches (besides EG-differences)?
Thank you!

barrus@uri.edu