Dense graphs and a conjecture for tree-depth

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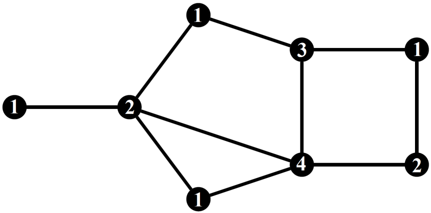
Joint work with John Sinkovic (University of Waterloo)

Tree-depth

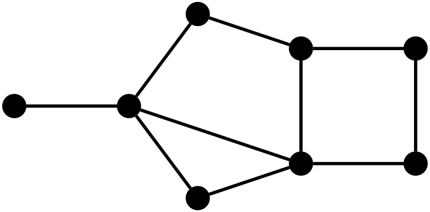
(aka (vertex) ranking number, ordered coloring number, ...)



Tree-depth td(G)**:** the smallest number of labels needed in a labeling of the vertices of G such that every path with equally-labeled endpoints also has a higher label. (Here, td(G) $= 4$)

Equivalently, the minimum number of vertex deletion steps needed to delete all of G, where in each step at most one vertex is deleted from each connected component.

Tree-depth and minors



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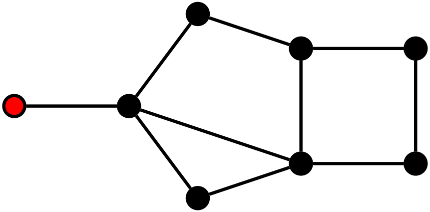

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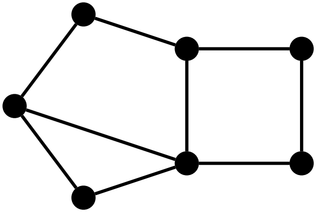

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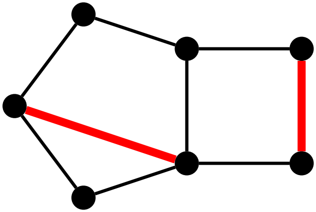

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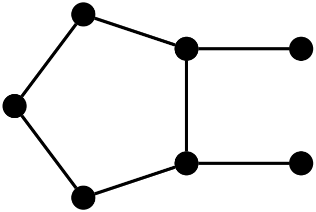

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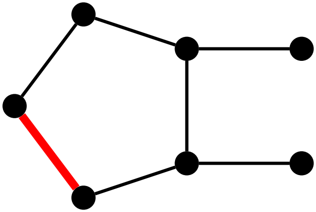

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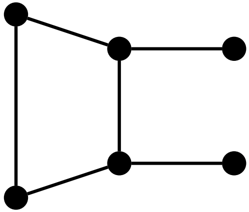

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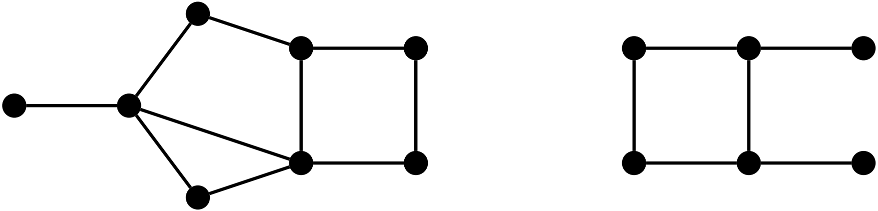
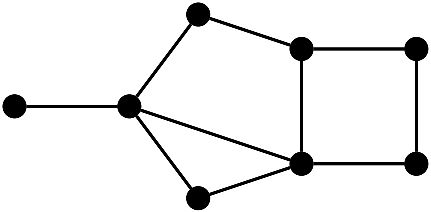
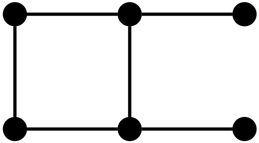


Theorem

If G contains H as a minor, then $\mathrm{td}(G) \geq \mathrm{td}(H)$.

Tree-depth and minors





Theorem

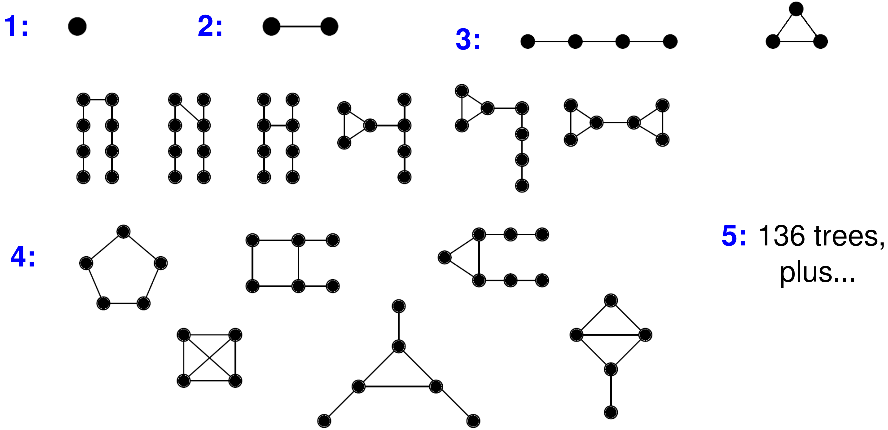
If G contains H as a minor, then $\text{td}(G) \geq \text{td}(H)$.

NOTE: A substructure of G forces G to have a higher tree-depth than it might otherwise have.

Call a graph **critical** if every proper minor has a smaller tree-depth. (k**-critical** = critical, with tree-depth k)

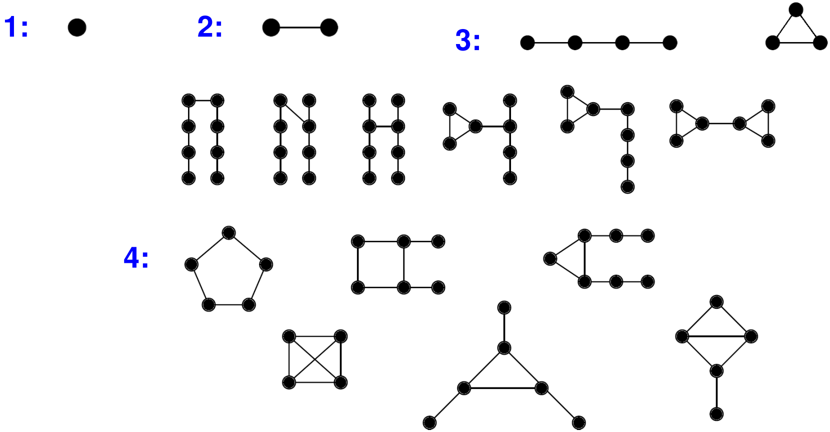
Critical graphs for small tree-depths

(Dvořák–Giannopoulou–Thilikos, '09, '12)



- What structural properties must critical graphs possess?

Structural properties of *k*-critical graphs

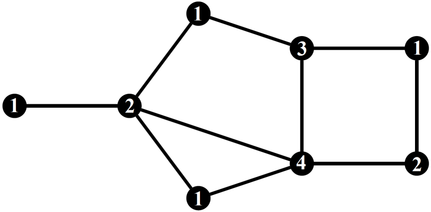


Conjectures If G is k-critical, then

- $|V(G)| \leq 2^{k-1}$ [Dvořák–Giannopoulou–Thilikos, ’09, ’12]
- $\Delta(G) \leq k - 1$ [B–Sinkovic, ’16]

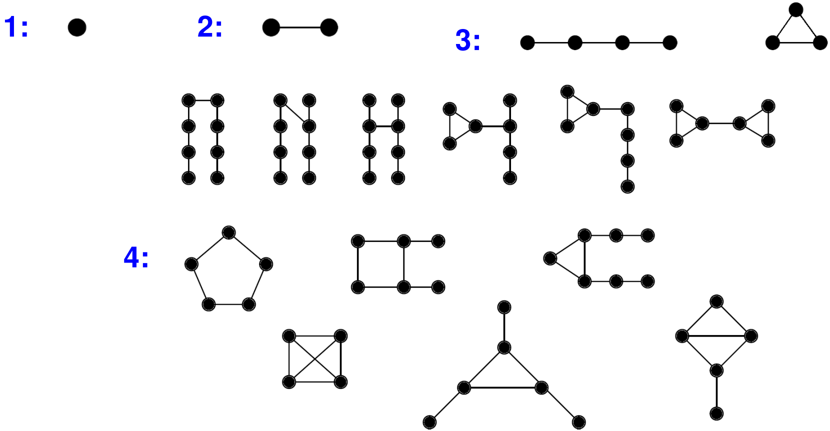
An approach to the max degree conjecture

Any vertex with the smallest label has neighbors with distinct labels.



If in some optimal labeling of G a vertex of maximum degree receives the label 1, then $\Delta(G) \leq \mathrm{td}(G) - 1$.

Stronger conjectures



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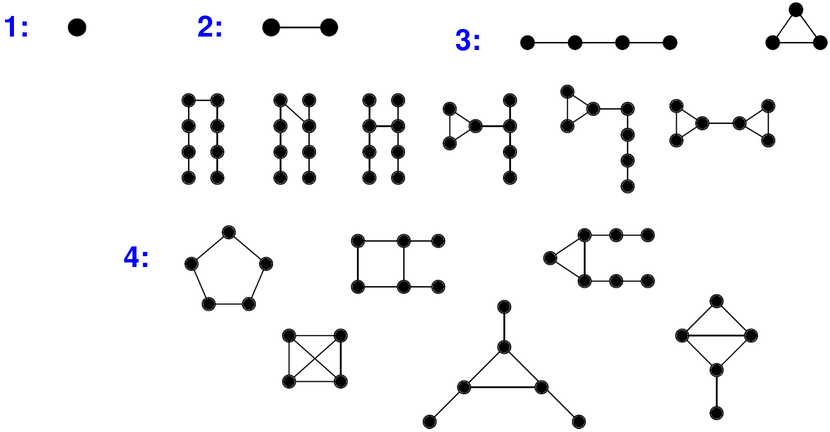
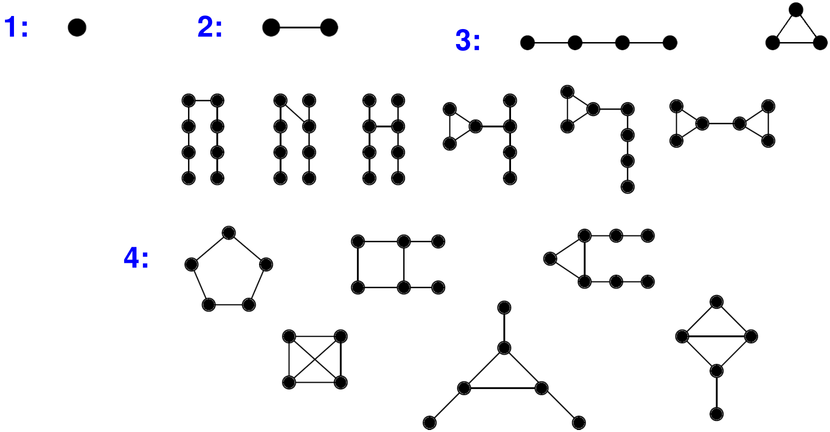


Conjectures If G is k-critical, then

- G has an optimal labeling where some vertex with maximum degree is labeled 1.

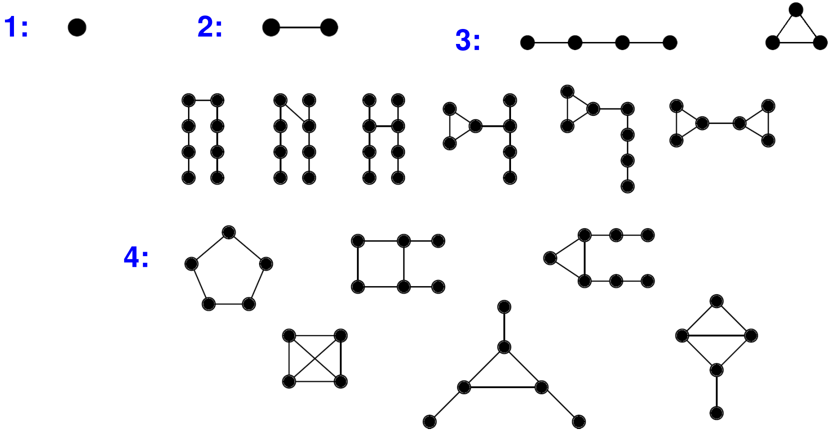
Stronger conjectures



Conjectures If G is k-critical, then

- For **each** vertex v of G, there is an optimal labeling where v receives label 1.

Stronger conjectures

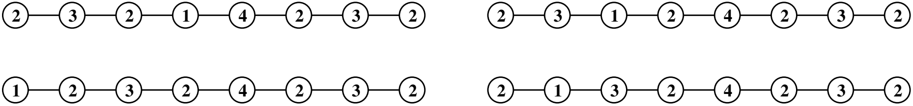


Conjectures If G is k-critical, then

- For **each** vertex v of G, there is an optimal labeling where v is the **unique** vertex receiving label 1.

1-uniqueness

Given a graph G, we say that a vertex v is **1-unique** if there is an optimal labeling of G where v is the only vertex receiving label 1.



We say a graph G is **1-unique** if each vertex of G is 1-unique.

Observation

If G is 1-unique, then $\Delta(G) \leq \text{td}(G) - 1$.

Conjecture

If G is critical, then G is 1-unique.

Plausibility: 1-uniqueness as a type of criticality?

Theorem (B–Sinkovic, 2016)

If G is a 1-unique graph, then

- *deleting any vertex of G lowers the tree-depth, and*
- *contracting any edge of G lowers the tree-depth.*

Hence G has a connected spanning subgraph that is $\mathrm{td}(G)$*-critical.*

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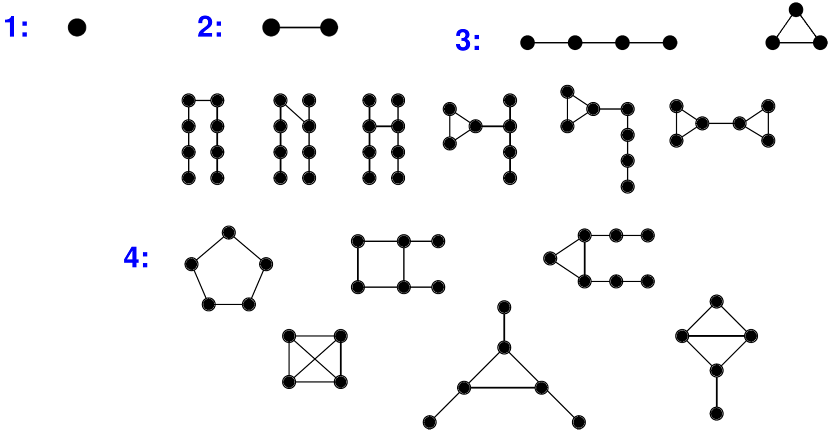
NOTE: We get a $\mathrm{td}(G)$***-critical graph by deleting <u>some</u> (possibly none) of the edges of*** G***.***

Plausibility: small tree-depths

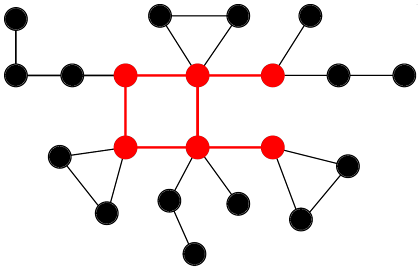


Conjecture

- If G is k-critical, then G is 1-unique.

True for $k \leq 4$...

Another application of 1-uniqueness

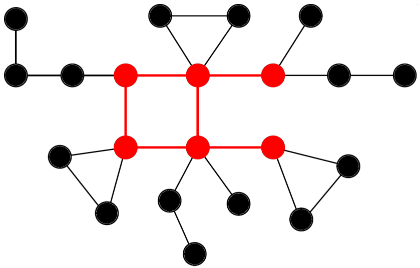


Theorem (B–Sinkovic, 2016)

Graphs G constructed with k-critical "appendages" L_i and an ℓ-critical "core" H...

- *...have tree-depth $k + \ell - 1$ if H and all the L_i are critical;*
- *...are critical if H is also 1-unique;*
- *...are 1-unique if H and all the L_i are 1-unique;*

Another application of 1-uniqueness

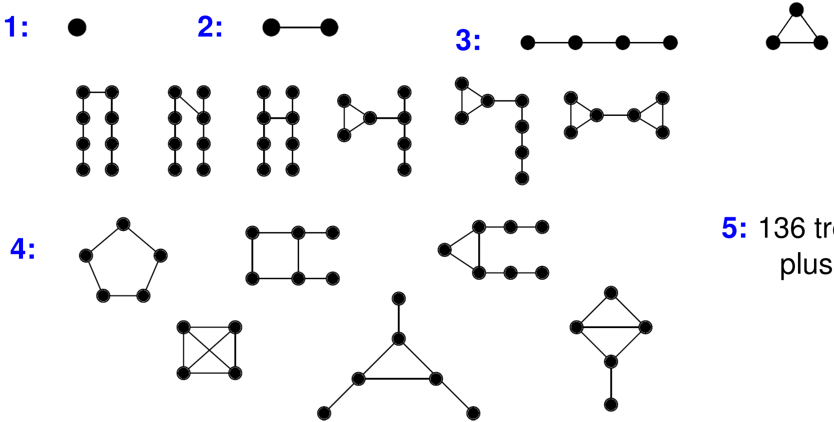


Theorem (B–Sinkovic, 2016)

Graphs G constructed with k-critical "appendages" L_i and an ℓ-critical "core" H...

- *...have order at most $2^{\mathrm{td}(G)-1}$ if $|V(H)| \leq 2^{\ell-1}$ and $|V(L_i)| \leq 2^{k-1}$ for all i.*

Some notes



5: 136 trees, plus...

Observations

- Max degree conjecture always true, graph order bound often true (at least!) if all critical graphs are 1-unique.

Some notes

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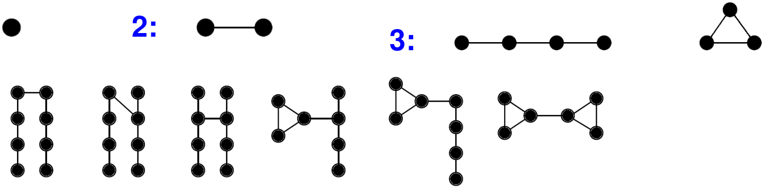

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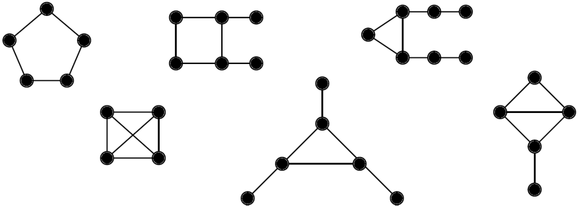


5: 136 trees, plus...

Observations

- Max degree conjecture always true, graph order bound often true (at least!) if all critical graphs are 1-unique.
- ***Most of the critical graphs seem to be sparse...***

Another view

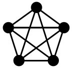
Of course, K_n is always dense, n-critical...

Curiosity: For which (critical?) graphs G is tree-depth close to $n(G)$?

High tree-depth

Let G be a graph with $n(G) = n$.

- $\text{td}(G) = n$ iff G is $\{2K_1\}$-free.



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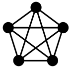

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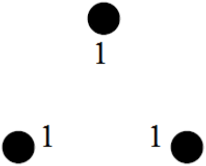
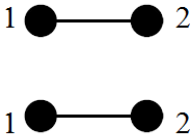


Label the $3K_1$ the same; distinct labels on all other vertices.

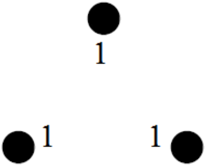
$$\text{td}(G) \leq 1 + (n - 3) = n - 2.$$



Label the $2K_2$ with two labels; distinct labels on all other vertices.

$$\text{td}(G) \leq 2 + (n - 4) = n - 2.$$

$\mathrm{td}(G) \geq n - 1$ iff G is $\{3K_1, 2K_2\}$-free.



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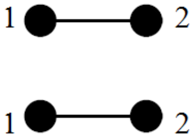

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NOTE: A substructure of G forces G to have a lower tree-depth than it might otherwise have.

$\mathrm{td}(G) \geq n - 1$ iff G is $\{3K_1, 2K_2\}$-free.

Low tree-depth forces induced subgraphs

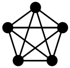
If $\mathrm{td}(G) \leq n - 2$ then fix an optimal labeling. Two vertices share labels with other vertices.

Case: Only one label appears on multiple vertices.

Case: Two labels appears on multiple vertices.

High tree-depth

Let G be a graph with $n(G) = n$.



- $\text{td}(G) = n$ iff G is $\{2K_1\}$-free.
- $\text{td}(G) \geq n - 1$ iff G is $\{3K_1, 2K_2\}$-free.
- $\text{td}(G) \geq n - 2$ iff G is $\{4K_1, 2K_2 + K_1, P_3 + K_2, 2K_3\}$-free.
- Conjectured: $\text{td}(G) \geq n - 3$ iff G is $\{5K_1, 2K_2 + 2K_1, P_3 + K_2 + K_1, 2K_3 + K_1, 2P_3, S_4 + K_2, 3K_2, \text{diamond} + K_3, C_4 + K_3, \text{paw} + K_3, P_4 + K_3, 2K_4\}$-free.

High tree-depth is hereditary

Theorem

Given a nonnegative integer k, let $\mathcal{F}_k$ denote the set of minimal elements, under the induced subgraph ordering, of all graphs H for which $n(H) - \text{td}(H) \geq k + 1$. For all graphs G, the graph G satisfies $\text{td}(G) \geq n(G) - k$ if and only if G is $\mathcal{F}_k$-free.

Back to criticality and 1-uniqueness

Theorem

If G is an n-vertex critical graph ***and*** $\text{td}(G) \geq n - 1$*, then G is 1-unique.*

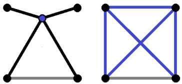
Back to criticality and 1-uniqueness

Theorem

*If G is an n-vertex critical graph **and** $\mathrm{td}(G) \geq n-1$, then G is 1-unique.*

Lemma

$\mathrm{td}(G) \geq n-1$ *iff* G *is* $\{3K_1, 2K_2\}$*-free.*



Theorem (B–Sinkovic, 2016)

A graph G is 1-unique if and only if every star-clique transformation lowers the tree-depth.

Back to criticality and 1-uniqueness

Theorem

If G is an n-vertex critical graph ***and*** $\mathrm{td}(G) \geq n-1$, *then G is 1-unique.*

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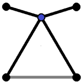
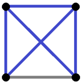




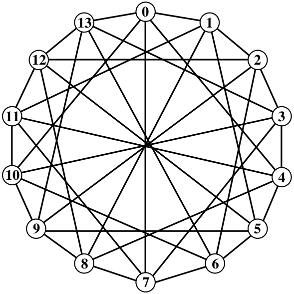
Theorem (B–Sinkovic, 2016)

A graph G is 1-unique if and only if every star-clique transformation lowers the tree-depth.

Outline: Easy to verify if $\mathrm{td}(G) = n$, so assume $\mathrm{td}(G) = n - 1$.
Let H be a graph obtained from G via a star-clique transformation at v.

- H is still $\{3K_1, 2K_2\}$-free.
- $\mathrm{td}(H) < \mathrm{td}(G)$ unless H is a complete graph.
- $G - vu$ has tree-depth $n - 2$; subgraphs induced force H to not be complete.

Interesting families of dense(ish) graphs



The **Andrasfai graphs**

And(k) has vertex set $\{0, \ldots, 3k-2\}$.

Edges join vertices whose difference is 1 modulo 3. (The graph is a circulant graph.)

- And(k) is triangle-free.
- Every maximal independent set is a vertex neighborhood.
- And(k) is k-connected, k-regular, and has independence number k.

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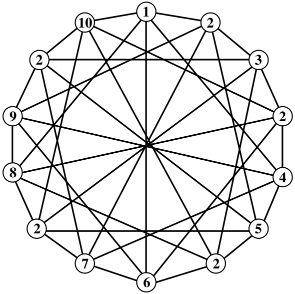

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- And(k) is triangle-free.
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- And(k) is k-connected, k-regular, and has independence number k.

- $\text{td}(\text{And}(k)) = 2k$ and $\text{td}(\text{And}(k) - v) = 2k - 1$
- And(k) and And(k) $- v$ are both 1-unique.
- And(k) and And(k) $- v$ are both tree-depth-critical.

Separating 1-uniqueness from criticality

Conjecture

If G is critical, then G is 1-unique.

Theorem (B–Sinkovic, 2016)

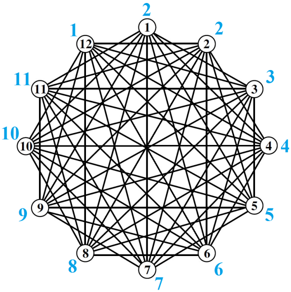
If G is a 1-unique graph, then

- *deleting any vertex of G lowers the tree-depth, and*
- *contracting any edge of G lowers the tree-depth.*

Hence G has a connected spanning subgraph that is $\text{td}(G)$-critical.

NOTE: We get a $\text{td}(G)$-critical graph by deleting some (possibly none) of the edges of G. **(How many?)**

Another dense family: cycle complements



For all $n \geq 5$,

$\text{td}\left(\overline{C_n}\right) = n - 1$ and $\overline{C_n}$ is 1-unique.

For $n \geq 8$, $\overline{C_n}$ is not critical.

But we can delete edges from $\overline{C_n}$ to reach an $(n-1)$-critical spanning subgraph...how many edges can it take?

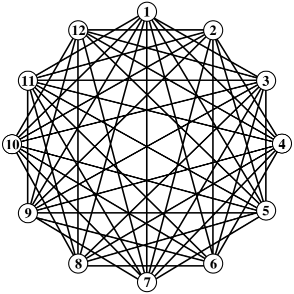
1-unique can be far from critical

For all $n \geq 5$, $\text{td}\left(\overline{C_n}\right) = n - 1$ and $\overline{C_n}$ is 1-unique.

For $n = 4k$ and $k \geq 2$, let G_n be obtained from $\overline{C_n}$ by deleting every other antipodal chord.

For $n = 4k \geq 8$, G_n is a $\{3K_1, 2K_2\}$-free spanning subgraph having k fewer edges than $\overline{C_n}$.

(Note that G_n need not be critical. Empirically, it seems that $\overline{C_n}$'s nearest spanning critical subgraph differs from it by an increasing number of edges.)



Resolution

Conjecture

For any k, if G is k-critical, then G is 1-unique.

Known true for

$k = 1, 2, 3, 4, \qquad n-1, n$

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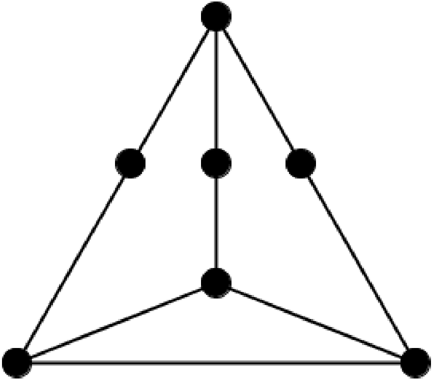
$$k = 1, 2, 3, 4, \qquad n-1, n$$

False for

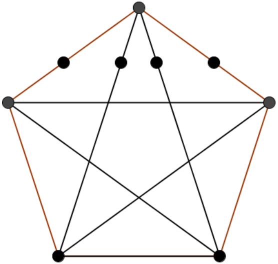
$$k = \qquad 5, 6, \ldots, n-2$$

Counterexample



A 7-vertex critical graph G with $\text{td}(G) = 5 = n(G) - 2$ that is not 1-unique!

Counterexamples

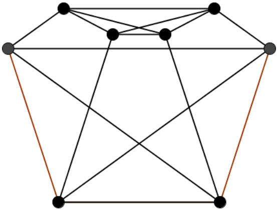


For $t \geq 2$, form H_t by subdividing all edges incident with a vertex of K_{t+2}.

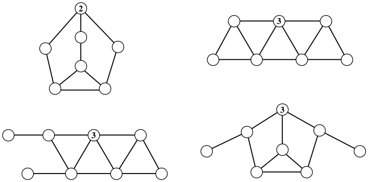
Here, $n(H_t) = 2t + 3$
$\text{td}(H_t) = t + 3 = n - t$

H_t is critical.

The vertex at the subdivided edges' center cannot receive the unique 1—a star-clique transformation at this vertex yields $K_{t+1} \Box K_2$, which has tree-depth $\lceil (3/2)(t+1) \rceil \geq t + 3$.



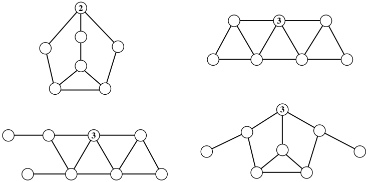
Finding counterexamples



Computer search using SageMath's graph database and functions

- Iterate through proper colorings of a graph.
- Identify colorings with a unique lowest label; identify 1-unique (or nearly 1-unique) graphs.
- Use 1-uniqueness to produce candidates for criticality testing.

Ongoing questions



- Using non-1-unique critical graphs to construct larger critical graphs.
- Empirically, it appears each k-critical non-1-unique graph yields a $(k-1)$-critical graph when the sole problem vertex is removed. Is this always the case?
- How to prove/disprove that critical graphs satisfy $\Delta(G) \leq \text{td}(G) - 1$?

Thank you!