Dense graphs and a conjecture for tree-depth

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Joint work with John Sinkovic (University of Waterloo)
Tree-depth \( td(G) \): the smallest number of labels needed in a labeling of the vertices of \( G \) such that every path with equally-labeled endpoints also has a higher label.  

(Here, \( td(G) = 4 \))

Equivalently, the minimum number of vertex deletion steps needed to delete all of \( G \), where in each step at most one vertex is deleted from each connected component.
Theorem

If $G$ contains $H$ as a minor, then $\text{td}(G) \geq \text{td}(H)$.
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\]

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Dense graphs and a conjecture for tree-depth

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Theorem

If $G$ contains $H$ as a minor, then $\text{td}(G) \geq \text{td}(H)$. 

NOTE: A substructure of $G$ forces $G$ to have a higher tree-depth than it might otherwise have.

Call a graph critical if every proper minor has a smaller tree-depth.

($k$-critical = critical, with tree-depth $k$)
Tree-depth and minors

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If $G$ contains $H$ as a minor, then $\text{td}(G) \geq \text{td}(H)$.

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Critical graphs for small tree-depths
(Dvořák–Giannopoulou–Thilikos, ’09, ’12)

1: ●
2: ●●
3: ●●●

4: 
5: 136 trees, plus...

What structural properties must critical graphs possess?
Structural properties of $k$-critical graphs

1: \hspace{1cm} 2: \hspace{1cm} 3: \hspace{1cm} 4:

Conjectures

If $G$ is $k$-critical, then

- $|V(G)| \leq 2^{k-1}$ \hspace{1cm} [Dvořák–Giannopoulou–Thilikos, ’09, ’12]
- $\Delta(G) \leq k - 1$ \hspace{1cm} [B–Sinkovic, ’16]
An approach to the max degree conjecture

Any vertex with the smallest label has neighbors with distinct labels.

If in some optimal labeling of $G$ a vertex of maximum degree receives the label 1, then $\Delta(G) \leq \text{td}(G) - 1$. 
Stronger conjectures

1:  

2:  

3:  

4:  

If $G$ is $k$-critical, then $G$ has an optimal labeling where some vertex with maximum degree is labeled 1.
Stronger conjectures

1:  2:  3:  4:

Conjectures

If $G$ is $k$-critical, then

- $G$ has an optimal labeling where some vertex with maximum degree is labeled 1.
Stronger conjectures

1: \[ \bullet \]
2: \[ \bullet - \bullet \]
3: \[ \bullet - \bullet - \bullet - \bullet \]
4: \[ \text{Various graphs showing conjectures} \]

Conjectures
If \( G \) is \( k \)-critical, then

- For each vertex \( v \) of \( G \), there is an optimal labeling where \( v \) receives label 1.
Stronger conjectures

1:  

2:  

3:  

4:  

Conjectures

If $G$ is $k$-critical, then

- For each vertex $v$ of $G$, there is an optimal labeling where $v$ is the unique vertex receiving label 1.
1-uniqueness

Given a graph $G$, we say that a vertex $v$ is **1-unique** if there is an optimal labeling of $G$ where $v$ is the only vertex receiving label 1.

We say a graph $G$ is **1-unique** if each vertex of $G$ is 1-unique.

**Observation**

If $G$ is 1-unique, then $\Delta(G) \leq \text{td}(G) - 1$.

**Conjecture**

If $G$ is critical, then $G$ is 1-unique.
Theorem (B–Sinkovic, 2016)

If $G$ is a 1-unique graph, then

- deleting any vertex of $G$ lowers the tree-depth, and
- contracting any edge of $G$ lowers the tree-depth.

Hence $G$ has a connected spanning subgraph that is $td(G)$-critical.
Plausibility: 1-uniqueness as a type of criticality?

**Theorem (B–Sinkovic, 2016)**

If $G$ is a 1-unique graph, then
- deleting any vertex of $G$ lowers the tree-depth, and
- contracting any edge of $G$ lowers the tree-depth.

Hence $G$ has a connected spanning subgraph that is $\text{td}(G)$-critical.

**NOTE:** We get a $\text{td}(G)$-critical graph by deleting some (possibly none) of the edges of $G$. 
Plausibility: small tree-depths

Conjecture

- If $G$ is $k$-critical, then $G$ is 1-unique.

True for $k \leq 4...$
Another application of 1-uniqueness

Graphs $G$ constructed with $k$-critical “appendages” $L_i$ and an $\ell$-critical “core” $H$...

- have tree-depth $k + \ell - 1$ if $H$ and all the $L_i$ are critical;
- are critical if $H$ is also 1-unique;
- are 1-unique if $H$ and all the $L_i$ are 1-unique;
Another application of 1-uniqueness

Theorem (B–Sinkovic, 2016)

Graphs $G$ constructed with $k$-critical “appendages” $L_i$ and an $\ell$-critical “core” $H$...

...have order at most $2^{\text{td}(G)-1}$ if $|V(H)| \leq 2^{\ell-1}$ and $|V(L_i)| \leq 2^{k-1}$ for all $i$. 

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Some notes

1: 

2: 

3: 

4: 

5: 136 trees, plus...

Observations

- Max degree conjecture always true, graph order bound often true (at least!) if all critical graphs are 1-unique.
1: \[ \bullet \]
2: \[ \bullet \bullet \]
3: \[ \bullet \bullet \bullet \]
4: \[ \bullet \bullet \bullet \bullet \]
5: \[ \bullet \bullet \bullet \bullet \]

Observations

- Max degree conjecture always true, graph order bound often true (at least!) if all critical graphs are 1-unique.
- Most of the critical graphs seem to be sparse...
Another view

Of course, $K_n$ is always dense, $n$-critical...

**Curiosity:** For which (critical?) graphs $G$ is tree-depth close to $n(G)$?
High tree-depth

Let $G$ be a graph with $n(G) = n$.

- $td(G) = n$ iff $G$ is $\{2K_1\}$-free.
High tree-depth

Let $G$ be a graph with $n(G) = n$.

- $\text{td}(G) = n$ iff $G$ is $\{2K_1\}$-free.
- $\text{td}(G) \geq n - 1$ iff $G$ is $\{3K_1, 2K_2\}$-free.
\( \text{td}(G) \geq n - 1 \) iff \( G \) is \( \{3K_1, 2K_2\}\)-free.

Label the \( 3K_1 \) the same; distinct labels on all other vertices.

\[
\text{td}(G) \leq 1 + (n - 3) = n - 2.
\]

Label the \( 2K_2 \) with two labels; distinct labels on all other vertices.

\[
\text{td}(G) \leq 2 + (n - 4) = n - 2.
\]
\( \text{td}(G) \geq n - 1 \) iff \( G \) is \( \{3K_1, 2K_2\}\)-free.

Label the 3\( K_1 \) the same; distinct labels on all other vertices.

\[ \text{td}(G) \leq 1 + (n - 3) = n - 2. \]

Label the 2\( K_2 \) with two labels; distinct labels on all other vertices.

\[ \text{td}(G) \leq 2 + (n - 4) = n - 2. \]

**NOTE:** A substructure of \( G \) forces \( G \) to have a **lower** tree-depth than it might otherwise have.
\( \text{td}(G) \geq n - 1 \) iff \( G \) is \( \{3K_1, 2K_2\}\)-free.

Low tree-depth forces induced subgraphs

If \( \text{td}(G) \leq n - 2 \) then fix an optimal labeling. Two vertices share labels with other vertices.

Case: Only one label appears on multiple vertices.

Case: Two labels appears on multiple vertices.
Let $G$ be a graph with $n(G) = n$.

- $\text{td}(G) = n$ iff $G$ is $\{2K_1\}$-free.

- $\text{td}(G) \geq n - 1$ iff $G$ is $\{3K_1, 2K_2\}$-free.

- $\text{td}(G) \geq n - 2$ iff $G$ is $\{4K_1, 2K_2 + K_1, P_3 + K_2, 2K_3\}$-free.

- Conjectured: $\text{td}(G) \geq n - 3$ iff $G$ is $\{5K_1, 2K_2 + 2K_1, P_3 + K_2 + K_1, 2K_3 + K_1, 2P_3, S_4 + K_2, 3K_2, \text{diamond} + K_3, C_4 + K_3, \text{paw} + K_3, P_4 + K_3, 2K_4\}$-free.
High tree-depth is hereditary

**Theorem**

Given a nonnegative integer $k$, let $\mathcal{F}_k$ denote the set of minimal elements, under the induced subgraph ordering, of all graphs $H$ for which $n(H) - \text{td}(H) \geq k + 1$. For all graphs $G$, the graph $G$ satisfies $\text{td}(G) \geq n(G) - k$ if and only if $G$ is $\mathcal{F}_k$-free.
Theorem

If $G$ is an $n$-vertex critical graph and $\text{td}(G) \geq n - 1$, then $G$ is 1-unique.

Lemma

$\text{td}(G) \geq n - 1$ iff $G$ is $\{3K_1, 2K_2\}$-free.

Theorem (B–Sinkovic, 2016)

A graph $G$ is 1-unique if and only if every star-clique transformation lowers the tree-depth.
Back to criticality and 1-uniqueness

**Theorem**

If $G$ is an $n$-vertex critical graph and $\text{td}(G) \geq n - 1$, then $G$ is 1-unique.

**Lemma**

$\text{td}(G) \geq n - 1$ iff $G$ is $\{3K_1, 2K_2\}$-free.

**Theorem (B–Sinkovic, 2016)**

A graph $G$ is 1-unique if and only if every star-clique transformation lowers the tree-depth.
Back to criticality and 1-uniqueness

**Theorem**

If $G$ is an $n$-vertex critical graph and $\text{td}(G) \geq n - 1$, then $G$ is 1-unique.

**Lemma**

$t_d(G) \geq n - 1$ iff $G$ is $\{3K_1, 2K_2\}$-free.

**Theorem (B–Sinkovic, 2016)**

A graph $G$ is 1-unique if and only if every star-clique transformation lowers the tree-depth.

**Outline:**

Easy to verify if $t_d(G) = n$, so assume $t_d(G) = n - 1$. Let $H$ be a graph obtained from $G$ via a star-clique transformation at $v$.

- $H$ is still $\{3K_1, 2K_2\}$-free.
- $t_d(H) < t_d(G)$ unless $H$ is a complete graph.
- $G - vu$ has tree-depth $n - 2$; subgraphs induced force $H$ to not be complete.
Interesting families of dense(ish) graphs

The **Andrasfai graphs**

$\text{And}(k)$ has vertex set $\{0, \ldots , 3k - 2\}$.

Edges join vertices whose difference is 1 modulo 3. (The graph is a circulant graph.)

- $\text{And}(k)$ is triangle-free.
- Every maximal independent set is a vertex neighborhood.
- $\text{And}(k)$ is $k$-connected, $k$-regular, and has independence number $k$. 
Interesting families of dense(ish) graphs

The **Andrasfai graphs**

And\((k)\) has vertex set \(\{0, \ldots, 3k - 2\}\).

Edges join vertices whose difference is 1 modulo 3. (The graph is a circulant graph.)

- And\((k)\) is triangle-free.
- Every maximal independent set is a vertex neighborhood.
- And\((k)\) is \(k\)-connected, \(k\)-regular, and has independence number \(k\).

\[
\begin{align*}
\text{td}(\text{And}(k)) &= 2k \\
\text{td}(\text{And}(k) - v) &= 2k - 1
\end{align*}
\]

- And\((k)\) and And\((k) - v\) are both 1-unique.
- And\((k)\) and And\((k) - v\) are both tree-depth-critical.
Conjecture

If $G$ is critical, then $G$ is 1-unique.

Theorem (B–Sinkovic, 2016)

If $G$ is a 1-unique graph, then

- deleting any vertex of $G$ lowers the tree-depth, and
- contracting any edge of $G$ lowers the tree-depth.

Hence $G$ has a connected spanning subgraph that is $td(G)$-critical.

**NOTE:** We get a $td(G)$-critical graph by deleting some (possibly none) of the edges of $G$.

(How many?)
Another dense family: cycle complements

For all $n \geq 5$, 
\[ \text{td} \left( \overline{C_n} \right) = n - 1 \] 
and $\overline{C_n}$ is 1-unique.

For $n \geq 8$, $\overline{C_n}$ is not critical.

But we can delete edges from $\overline{C_n}$ to reach an $(n - 1)$-critical spanning subgraph...how many edges can it take?
For all $n \geq 5$, $\text{td}(\overline{C_n}) = n - 1$ and $\overline{C_n}$ is 1-unique.

For $n = 4k$ and $k \geq 2$, let $G_n$ be obtained from $\overline{C_n}$ by deleting every other antipodal chord.

For $n = 4k \geq 8$, $G_n$ is a $\{3K_1, 2K_2\}$-free spanning subgraph having $k$ fewer edges than $\overline{C_n}$.

(Note that $G_n$ need not be critical. Empirically, it seems that $\overline{C_n}$’s nearest spanning critical subgraph differs from it by an increasing number of edges.)
Conjecture

For any $k$, if $G$ is $k$-critical, then $G$ is 1-unique.

Known true for

$$k = 1, 2, 3, 4,$$  \hspace{1cm}  $$n - 1, n$$
Resolution

Conjecture
For any $k$, if $G$ is $k$-critical, then $G$ is 1-unique.

Known true for
\[ k = 1, 2, 3, 4, \quad n - 1, n \]

False for
\[ k = 5, 6, \ldots, n - 2 \]
A 7-vertex critical graph $G$ with $\text{td}(G) = 5 = n(G) - 2$ that is not 1-unique!
For $t \geq 2$, form $H_t$ by subdividing all edges incident with a vertex of $K_{t+2}$.

Here, $n(H_t) = 2t + 3$

$td(H_t) = t + 3 = n - t$

$H_t$ is critical.

The vertex at the subdivided edges’ center cannot receive the unique 1—a star-clique transformation at this vertex yields $K_{t+1} \square K_2$, which has tree-depth $\lceil (3/2)(t + 1) \rceil \geq t + 3$. 
Computer search using SageMath’s graph database and functions
- Iterate through proper colorings of a graph.
- Identify colorings with a unique lowest label; identify 1-unique (or nearly 1-unique) graphs.
- Use 1-uniqueness to produce candidates for criticality testing.
Ongoing questions

- Using non-1-unique critical graphs to construct larger critical graphs.

- Empirically, it appears each \( k \)-critical non-1-unique graph yields a \((k - 1)\)-critical graph when the sole problem vertex is removed. Is this always the case?

- How to prove/disprove that critical graphs satisfy \( \Delta(G) \leq \text{td}(G) - 1 \)?
Thank you!