Independence number and the Havel–Hakimi Residue

Michael D. Barrus

URI Discrete Math Seminar
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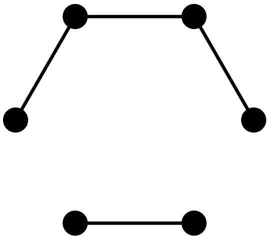
The independence number



The Havel–Hakimi Algorithm

Delete, reduce, reorder

Degree sequence?

$$d = (2, 2, 1, 1, 1, 1)$$

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$$d = (\mathbf{2}, \underline{2}, \underline{1}, 1, 1, 1)$$
$$d^1 = \quad (1, 0, 1, 1, 1)$$

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$$d^1 = \quad (1, 1, 1, 1, 0)$$

Theorem (V. Havel, 1995; S.L. Hakimi, 1962)

d is the degree sequence of a simple graph if and only if d^1 is.

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The **residue** $R(d)$ or $R(G)$ is the number of zeroes remaining at the end.

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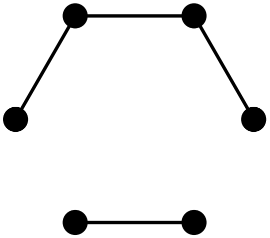

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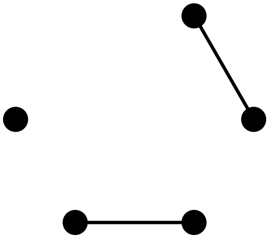

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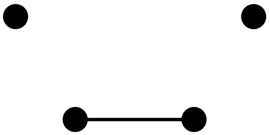

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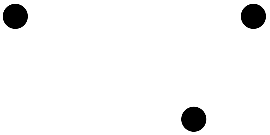

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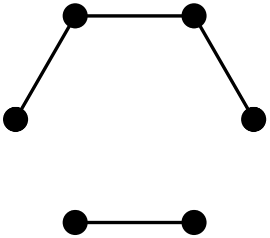

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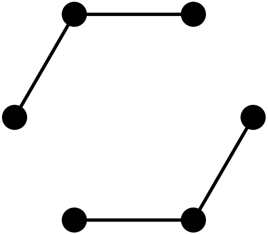

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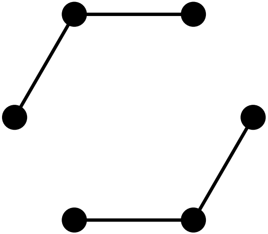

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For all graphs G, $R(G) \leq \alpha(G)$.

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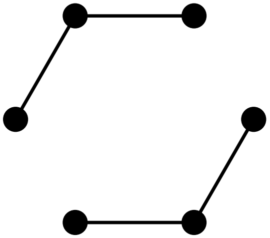

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Conjectured originally by Fajtlowicz' computer program Graffiti; improvements to the proof provided by Griggs and Kleitman (1994) and Triesch (1996).

Big Questions

- **How tight is the** $R(G) \leq \alpha(G)$ **bound?**

- **For which graphs** G **does** $R(G) = \alpha(G)$**?**

How tight is the $R(G) \leq \alpha(G)$ bound?

- One of the tightest known lower bounds
  - rigorously: (Favaron et al., '91) outperforms Brook's, Turan's, Hansen's, Caro–Wei's
  - anecdotally: (Nelson, '01); (Willis, '11) outperforms Wilf's bound?
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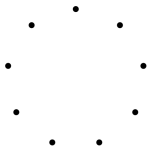
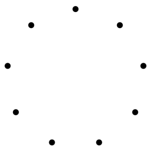
$$R(d) \leq \alpha_{\text{min}}(d) = \alpha(G) \leq \cdots \leq \alpha(G')$$

How tight is the $R(G) \leq \alpha_{\min}(d)$ bound?

- The inequality may be proper:

$$d = (4,4,4,4,4,4,4,4,4) \qquad R(d) = 2 \qquad \alpha_{\min}(d) = 3$$





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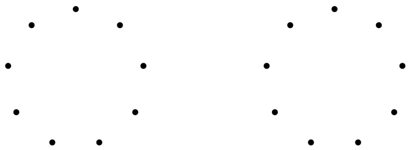

- **Theorem** (Nelson–Radcliffe, 2004)

  If d is **semi-regular**, then

$$R(d) \leq \alpha_{\min}(d) \leq R(d) + 1,$$

  and we know which d are which.

How tight is the $R(G) \leq \alpha(G)$ bound?

An idea:

$$R(d) \leq \alpha_{\min}(d) = \underbrace{\alpha(G) \leq \cdots \leq \alpha(G')}_{\text{What if this can't be large?}}$$

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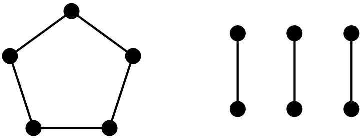
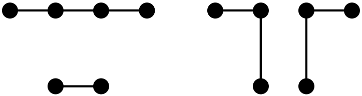
A **unigraph** is a graph that is the **unique realization** (up to isomorphism) of its degree sequence.



UNIGRAPHS



NOT UNIGRAPHS

How tight is the $R(G) \leq \alpha(G)$ bound?

Theorem (B, 2012)

If G is a ***unigraph****, then*

$$R(G) \leq \alpha(G) \leq R(G) + 1,$$

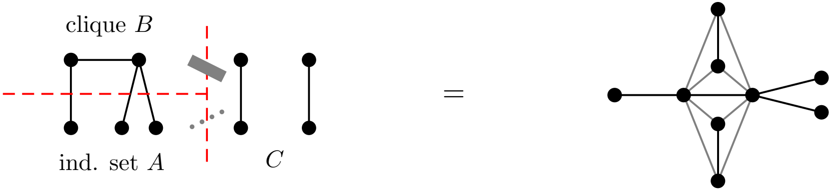
and we know which G are which.

(In fact, $R < \alpha$ in only one simple family of counterexamples.)

Key ideas of the proof

If G is a **unigraph**, then $R(G) \leq \alpha(G) \leq R(G) + 1$.

- R. Tyshkevich ('00) studied graph compositions of the form



She characterized unigraphs in terms of indecomposable components; these included C_5 and 6 infinite families of non-split graphs, and K_1 and 16 infinite families of split graphs.

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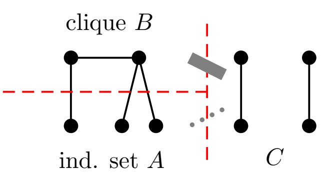

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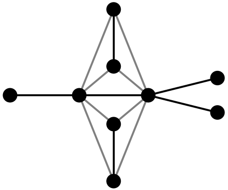

$(3, 2; 1, 1, 1) \circ (1, 1, 1, 1)$ $(7, 6, 3, 3, 3, 3, 1, 1, 1)$

Key ideas of the proof

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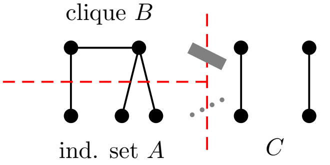
Lemma (B, 2012)

For a graph $G = (G_1, A, B) \circ G_0$, both

$$\alpha(G) = |A| + \alpha(G_0) \qquad \textit{and} \qquad R(G) = |A| + R(G_0).$$



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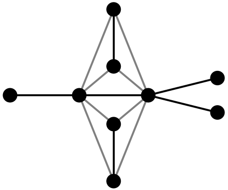

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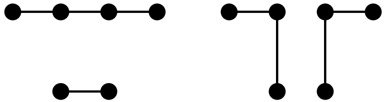
Mimicking vertex deletions in hopes of $R(G) = \alpha(G)$

Joint with Grant Molnar (BYU)



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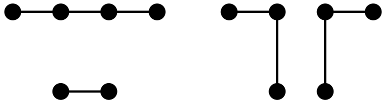

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A graph has the **strong Havel–Hakimi property** if in **every** induced subgraph **every** vertex of maximum degree has neighbors with as high of degrees as possible.

Let $\mathcal{S}$ be the class of all such graphs.

Results (B, Molnar, 2016)

$\mathcal{S}$ = {graphs with strong Havel–Hakimi property}

- All graphs in $\mathcal{S}$ can be constructed via the natural "reverse Havel–Hakimi process."

$$(0,0,0) \to (1,1,0,0) \to (1,1,1,1,0) \to (2,2,1,1,1,1)$$

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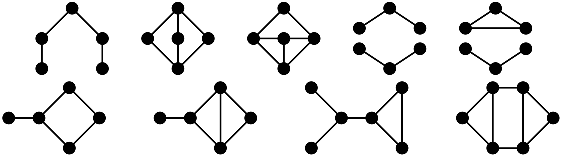

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- Maxine produces a maximum independent set for arbitrary graphs H whenever H is $\{C_4, P_5\}$-free.

Hereditary classes of graphs for which $R = \alpha$

(B, 2012, 2013; B, Molnar, 2016)

- **Split graphs**
- **Hereditary unigraphs**
- $\mathcal{S}$ (⊃ matrogenic graphs, matroidal graphs, threshold graphs)

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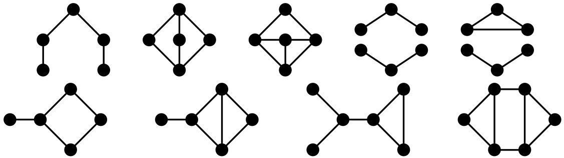

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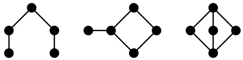

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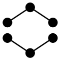

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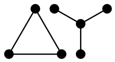

No strong patterns

Hereditary classes of graphs for which $R = \alpha$

(B, 2012, 2013; B, Molnar, 2016)

- **Split graphs**
- **Hereditary unigraphs**
- $\mathcal{S}$ (⊃ matrogenic graphs, matroidal graphs, threshold graphs)

Let $\mathcal{H}$ be the maximal hereditary family consisting of graphs G for which $R(G) = \alpha(G)$.

Minimal forbidden induced subgraphs for $\mathcal{H}$?

Vertices	5	6	7	8	9	10
Subgraphs	3	1	1	8	19	

...

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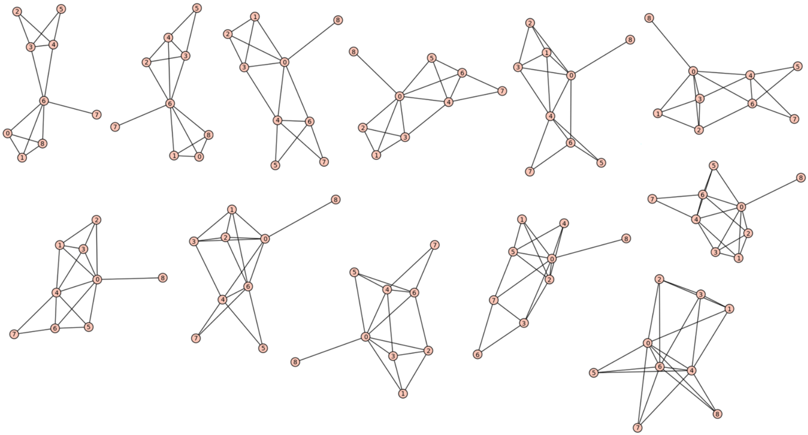
Remaining Questions

- **How tight is the $R(G) \leq \alpha(G)$ bound?**

- **For which graphs G does $R(G) = \alpha(G)$?**

Some forbidden subgraphs for $\mathcal{H}$ with 9 vertices



Thank you!