The A_4-structure of a graph

Michael D. Barrus

Department of Mathematics
Brigham Young University

Graphs and Matrices Seminar
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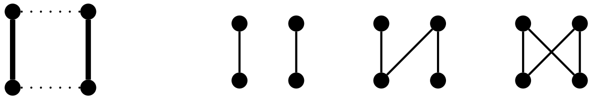
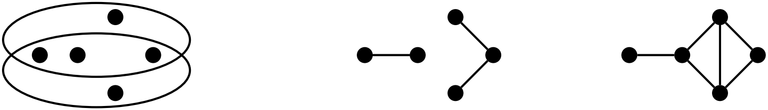
The A_4-Structure

Alternating 4-cycle (A_4)



A_4-structure H of a graph G

$$V(H) = V(G), \qquad E(H) = \{A \subseteq V(G) : G[A] \cong 2K_2 \text{ or } C_4 \text{ or } P_4\}$$



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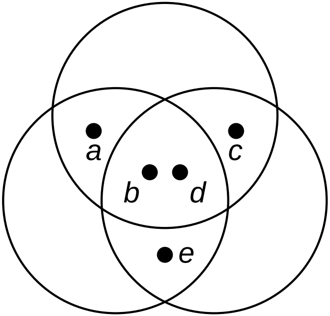
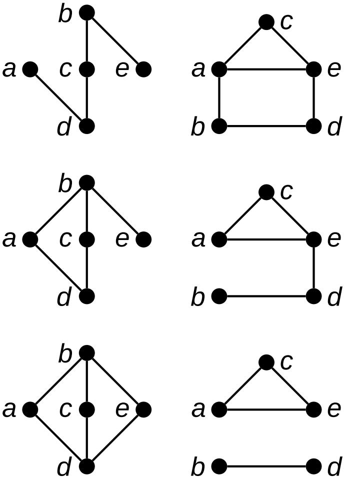
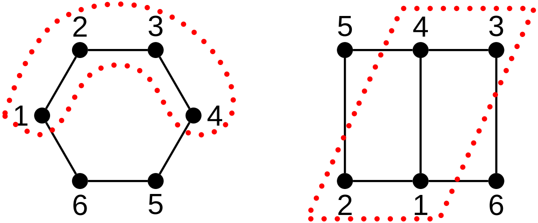




The P_4-Structure of a Graph

Chvátal, 1984

$$V(H) = V(G), \quad E(H) = \{A \subseteq V(G) : G[A] \cong P_4\}$$



Theorem (Reed, 1987)

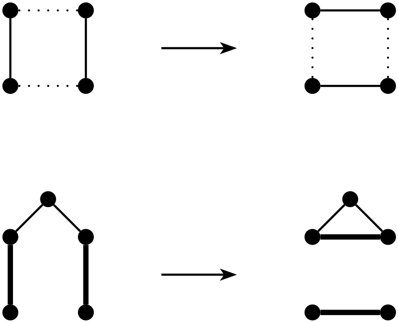
Let G and H be two graphs with isomorphic P_4-structures. Then G is perfect if and only if H is perfect.

P_4-Classes

Reprinted from A. Brandstädt and V. B. Le, Split-perfect graphs: characterization and algorithmic use, SIAM J. Discrete Math. 17(3), 341-360.

Motivation: Degree Sequences

2-switches



Theorem (Fulkerson–Hoffman–McAndrew, 1965)

$\deg(G) = \deg(H)$ *iff 2-switches transform G into H.*

Are there any A_4-structure/degree sequence connections?

Motivation: Graph Classes

- Threshold graphs
  $\{2K_2, C_4, P_4\}$-free
- Matrogenic graphs
  Vertex sets of A_4's are circuits of a matroid on V.
- Matroidal graphs
  Edge sets of A_4's are circuits of a matroid on E.

Motivation: Graph Classes

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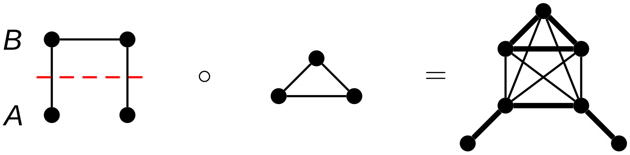
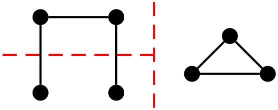
Can the A_4-structure be used to characterize other interesting classes?

A Graph Operation

Definition (Tyshkevich–Chernyak, 1978).

Given a split graph G with stable set A and clique B, and an arbitrary graph H, define the *composition* $(G, A, B) \circ H$ to be graph formed by adding to $G + H$ the edges in $\{uv : u \in B, v \in V(H)\}$.





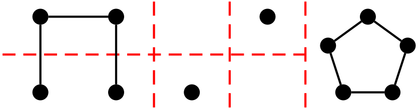
Canonical Decomposition

Theorem (Tyshkevich–Chernyak, 1978; Tyshkevich, 2000)

Every graph F can be represented as a composition

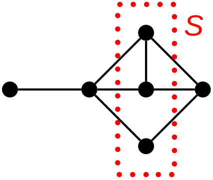
$$F = (G_k, A_k, B_k) \circ \cdots \circ (G_1, A_1, B_1) \circ F_0$$

of indecomposable components. Here the (G_i, A_i, B_i) are indecomposable splitted graphs and F_0 is an indecomposable graph. This decomposition is unique up to isomorphism of components.



Modules and P_4s

Definition.
A *module* is a vertex subset S such that each vertex outside S either dominates S or is isolated from S.



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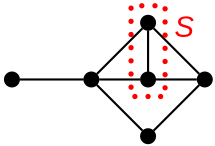

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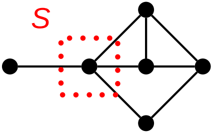

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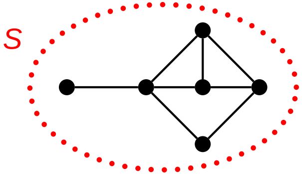

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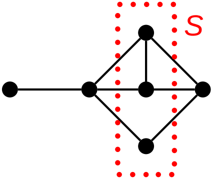

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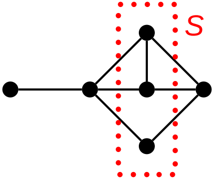
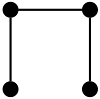




Theorem

- *An induced P_4 intersects a module in exactly 0, 1, or 4 vertices.*
- (Seinsche, 1974) *In a graph G every induced subgraph on at least 3 vertices contains a nontrivial module iff G is P_4-free.*

P_4-Structures and Decomposition

Primeval Decomposition Theorem (Jamison–Olariu, 1995)

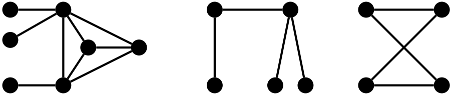
For any graph $G = (V, E)$ precisely one of the following conditions holds:

(i) *G is disconnected.*

(ii) *$\overline{G}$ is disconnected.*

(iii) *The P_4-structure of G is connected.*

(iv) *There exists a P_4-component hooked up to the rest of G in a special way.*



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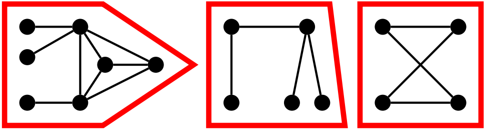

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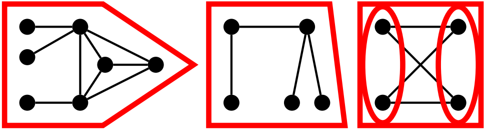

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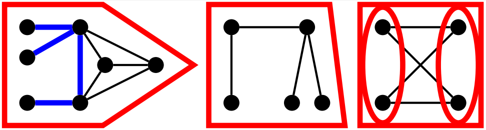

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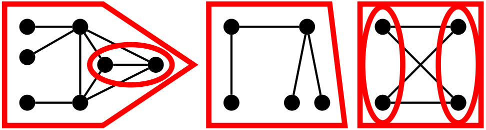

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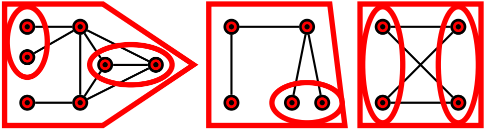

A_4-Analogues

Definition.
A *module* S is a vertex subset such that no alternating path of length 2 begins and ends in S and has its midpoint outside S.

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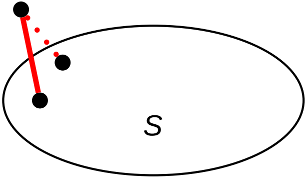

A_4-Analogues

Definition.

Define a ***strict*** *module* to be a vertex subset S such that no (**possibly closed**) alternating path of length 2 **or 3** begins and ends in S and has its midpoints outside S.

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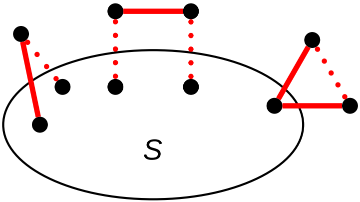

This is equivalent to not having alternating paths of *any* length begin and end in S.

A_4-Analogues

Proposition

An A_4 intersects a strict module in exactly 0 or 4 vertices.

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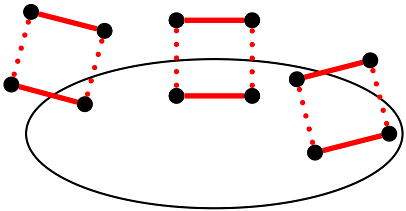

Proposition

In a graph G every induced subgraph on at least 2 vertices has a nontrivial strict module if and only if G is A_4-free, i.e., threshold.

Strict Modules and Graph Structure

Proposition

The vertices which dominate a strict module form a clique, and those which are nonadjacent to the strict module form an independent set.

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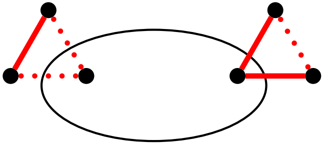

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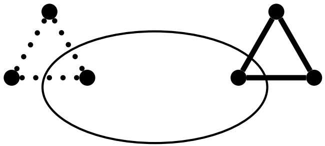

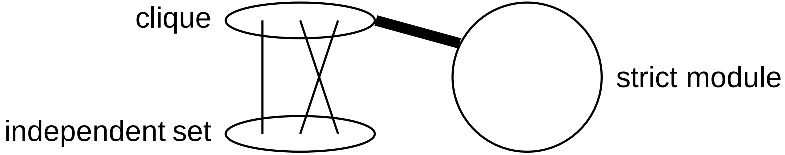

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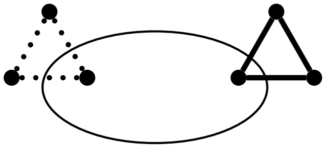

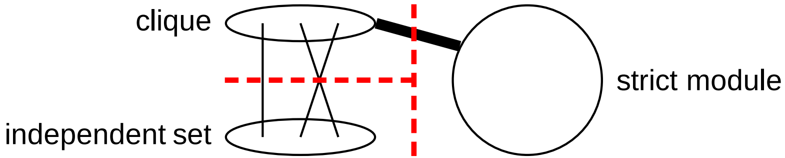

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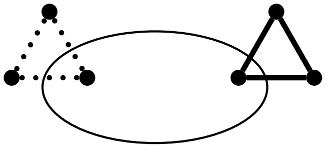
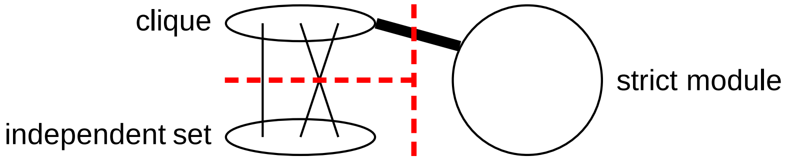


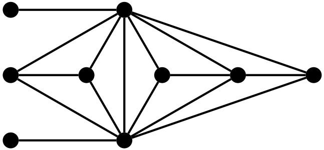


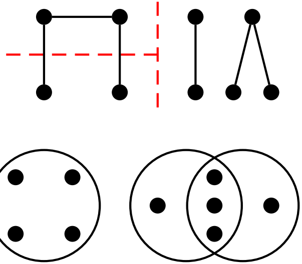
"Strict modular decomposition" = canonical decomposition

Indecomposable Graphs

Theorem

A graph is indecomposable in the canonical decomposition if and only if its A_4-structure is connected.





Proof later?

A Degree Sequence Connection

Theorem (Tyshkevich, 1980?, 2000)

An n-vertex graph with degree sequence d is decomposable if and only if there exists nonnegative integers p and q such that

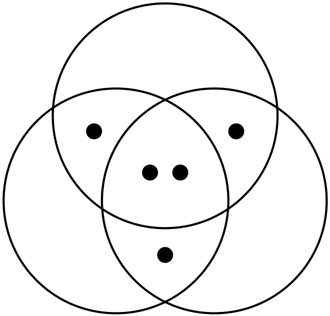
$$0 < p + q < n, \qquad \sum_{i=1}^{p} d_i = p(n-q-1) + \sum_{i=n-q+1}^{n} d_i.$$

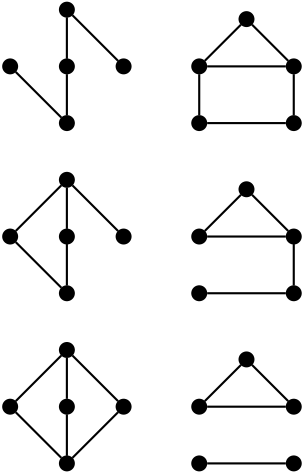
Corollary

If two graphs have the same degree sequence, then their A_4-structures have the same number and orders of components.

Obtaining All Realizations

Given an A_4-structure, how do we generate all graphs realizing it?





Obtaining Other Realizations: Decomposable Graphs

Substitutions and transpositions preserve A_4-structure.

The rightmost A_4-component may only be transposed if it has a split realization.

Which graphs have the same A_4-structure as a split graph?

A_4-Separable Graphs

Observation

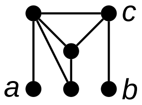
A graph G is A_4-split, i.e., it has the same A_4-structure as a split graph, iff each of its indecomposable component is A_4-split.

A graph G is A_4-separable if we can partition $V(G)$ into two sets so that each A_4 can be drawn with both edges and both nonedges spanning the divide.

A_4-separable $\implies$ A_4-split

A_4-Balanced Graphs with the BRP

A graph G is A_4-balanced if we can partition $V(G)$ into two sets so that each set contains two vertices of each A_4. An A_4-balanced graph has the bipartite restriction property if for each v, the graph G_v is bipartite.



G_a G_b G_c

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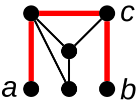
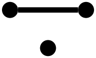


G_a



G_b

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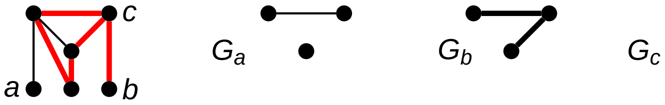

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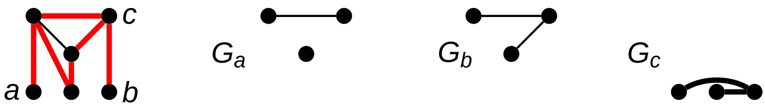

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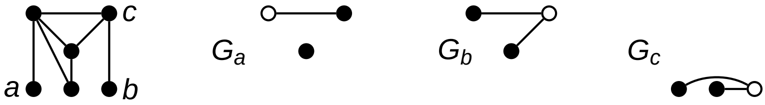
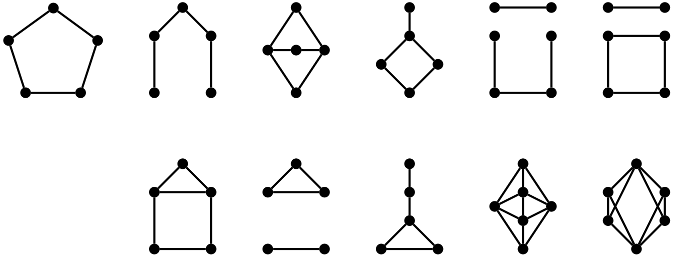


A_4-split $\implies$ A_4-balanced with the bipartite restriction property

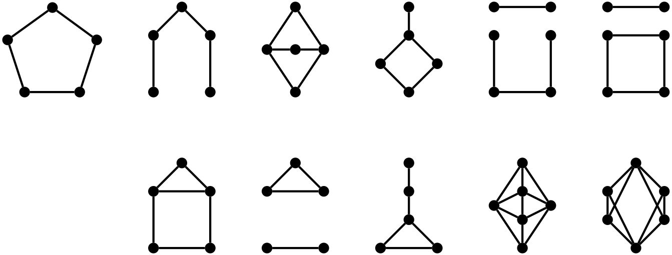
Forbidden subgraphs

The following graphs are not A_4-balanced or do not have the BRP:



Forbidden subgraphs

Say G induces none of the forbidden graphs:



G disconnected $\implies$ Each component is $\{K_3, C_4, P_4\}$-free

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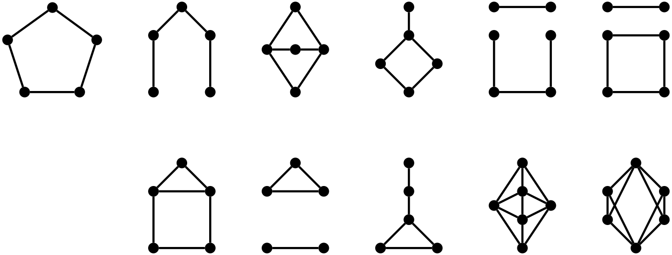

G disconnected $\implies$ Each component is a star

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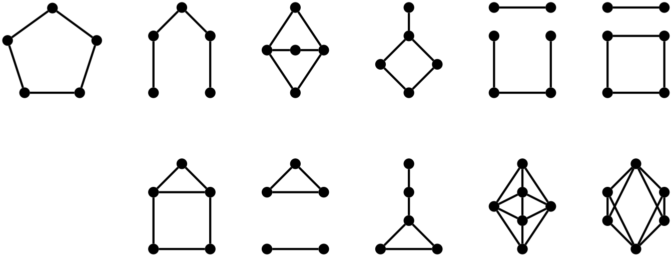
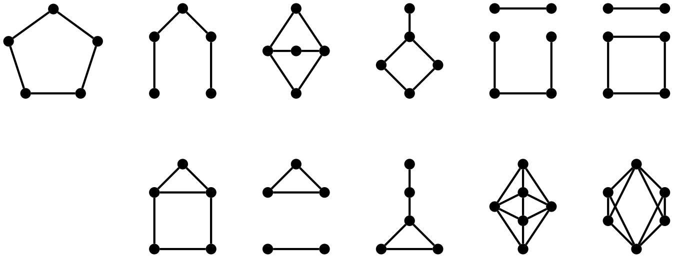


G connected, co-connected $\implies$ G is split.

Forbidden subgraphs

Say G induces none of the forbidden graphs:

$\mathcal{F}$



G A_4-balanced, has BRP $\implies$ G is $\mathcal{F}$-free $\implies$ G is split, or G or $\overline{G}$ is a forest of stars

Completing the chain

G split, or G or $\overline{G}$ a forest of stars $\implies$ G A_4-separable

A_4-Split Graphs

Theorem

For an indecomposable graph G with A_4-structure H, the following are equivalent:

(i) *G is A_4-split.*

(ii) *H is balanced and satisfies the bipartite restriction property.*

(iii) *G is $\{C_5, P_5, house, K_2 + K_3, K_{2,3}, P, \overline{P}, K_2 + P_4, P_4 \vee 2K_1, K_2 + C_4, 2K_2 \vee 2K_1\}$-free.*

(iv) *G is split, or G or $\overline{G}$ is a disjoint union of stars.*

(v) *G is A_4-separable.*

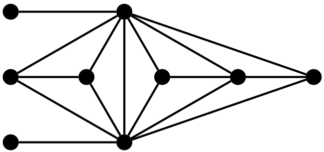
Left to Do

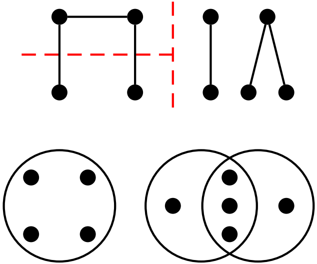
- Graph classes, especially A_4- and P_4-balanced graphs, and the A_4-analogues of the (q,t) graphs (Threshold $=(4,0)$, matroidal $=(5,1)$, ...).
- Other A_4-structure characteristics dependent only on degree sequence.
- Complete list of operations which suffice to link all realizations of an A_4-structure.

Appendix

Theorem

A graph is indecomposable in the canonical decomposition if and only if its A_4-structure is connected.



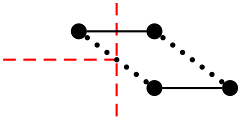


Beginnings

Lemma

The graphs $2K_2$, C_4, and P_4 are all indecomposable. Therefore, connected A_4-structure $\implies$ indecomposable.

Forbidden:



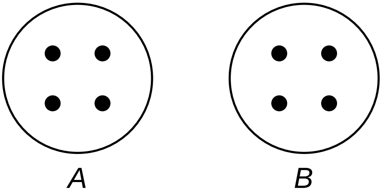
Lemma

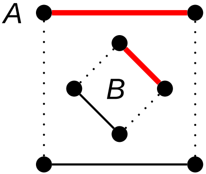
In an indecomposable graph G with more than 1 vertex, every vertex belongs to an alternating 4-cycle.

Disjoint A_4s

Lemma

If A and B are disjoint alternating 4-cycles in G such that no third alternating cycle in G intersects each, then either A induces P_4, with its interior vertices dominating B and the endpoints isolated from B (denote this by $A \to B$), or vice versa.





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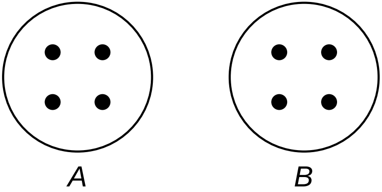

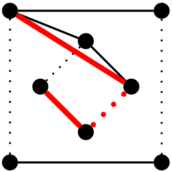

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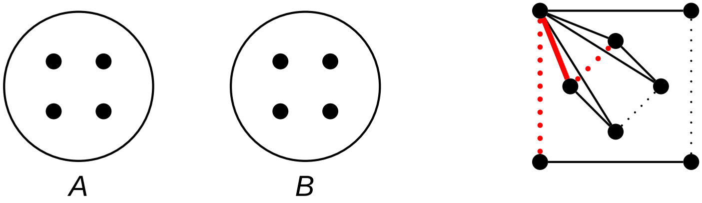

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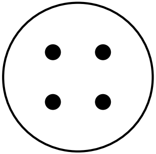
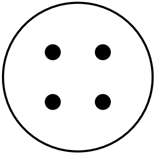
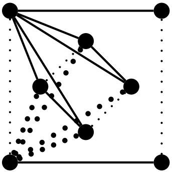


A



B



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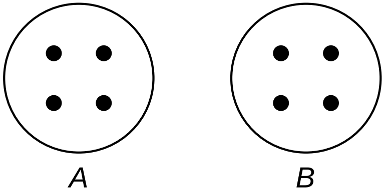

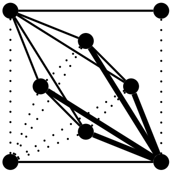

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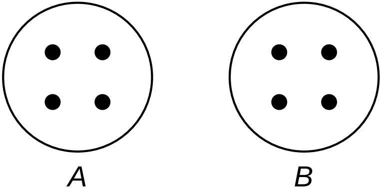

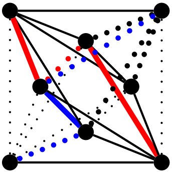

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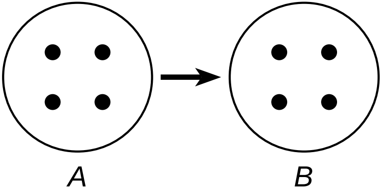
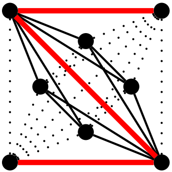
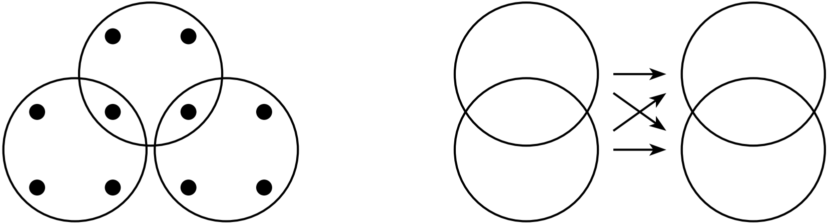




More on Disjoint A_4s

Corollary

Any two vertices which both belong to induced $2K_2$'s or C_4's have distance at most 3 in the A_4-structure.



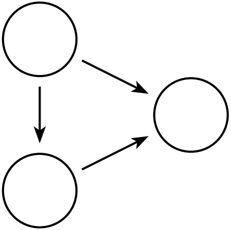
Lemma

The $\rightarrow$ relation is consistent among pairs of A_4s from different components of the A_4-structure.

Putting It All Together

Lemma

The $\rightarrow$ tournament on the A_4-components of a graph is acyclic.



Having a source implies the graph is decomposable.

$$\therefore \text{ not } A_4\text{-connected} \implies \text{decomposable.}$$

Return to talk

Putting It All Together

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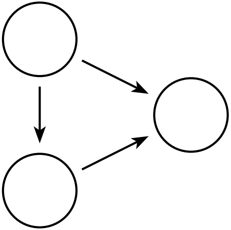

Having a source implies the graph is decomposable.

$$\therefore A_4\text{-connected} \iff \text{indecomposable.}$$

Return to talk