The Reconstruction Conjecture and Decks with Marked Cards

Michael D. Barrus

Department of Mathematics
Brigham Young University

Graphs and Matrices Seminar
September 5, 2012

The Graph Reconstruction Conjecture

aka Ulam's Conjecture, The Kelly–Ulam Conjecture

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- P. J. Kelly, On isometric transformations. Ph.D. Thesis, University of Wisconsin (1942).
- Kelly, P. J., A congruence theorem for trees, Pacific J. Math. 7 (1957), 961-968.
- Ulam, S. M., A collection of mathematical problems, Wiley, New York, 1960.

The Graph Reconstruction Conjecture

aka Ulam's Conjecture, The Kelly–Ulam Conjecture

> ***It is natural to wonder if any two graphs must be isomorphic when they have the same composition in terms of $(n-1)$-point subgraphs.***
>
> — *P. J. Kelly*

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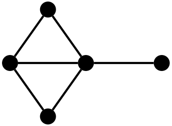
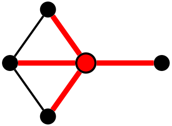
Graphs, cards, and decks



Graphs, cards, and decks



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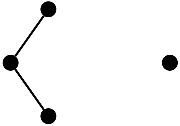
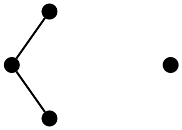
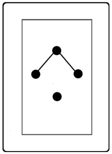


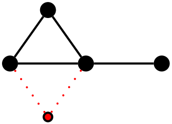
Graphs, cards, and decks

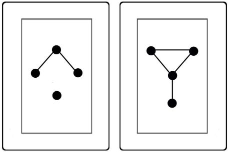




The cards

Graphs, cards, and decks





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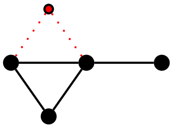
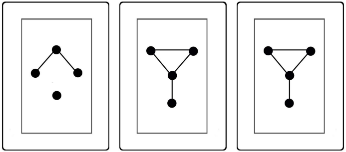
Graphs, cards, and decks





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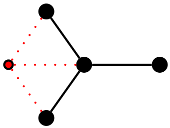
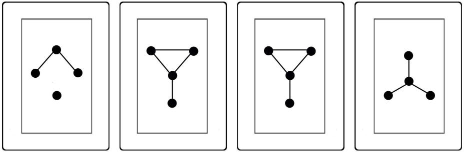
Graphs, cards, and decks





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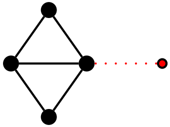
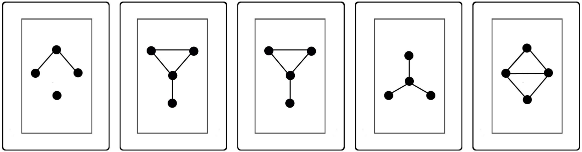
Graphs, cards, and decks





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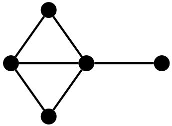
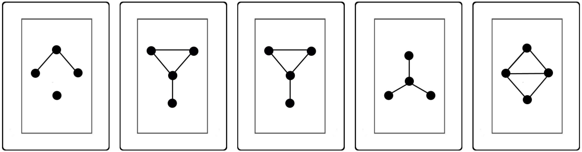
Graphs, cards, and decks





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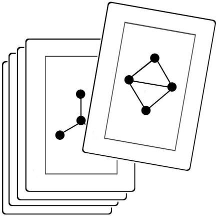

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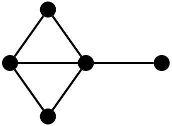
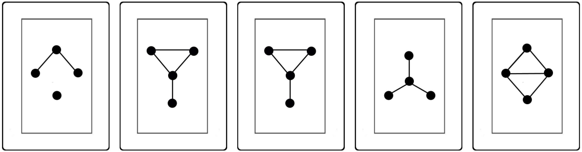
Graphs, cards, and decks





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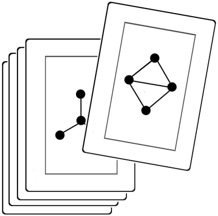


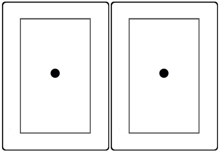
The deck

Graph Reconstruction Conjecture

No two (different) graphs on at least 3 vertices have the same deck.

*Given a deck, there is **only one** reconstruction.*

Examples



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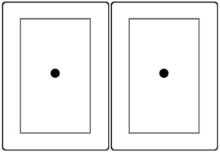

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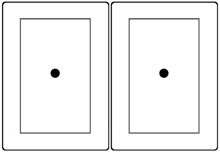

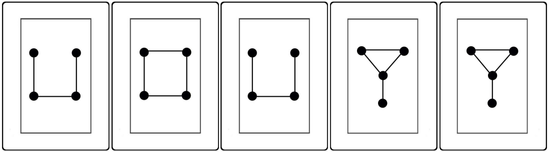

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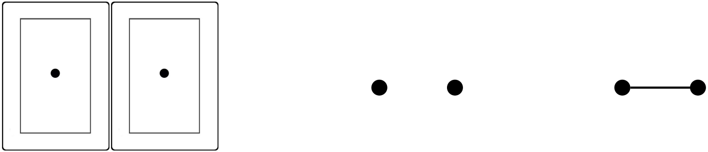

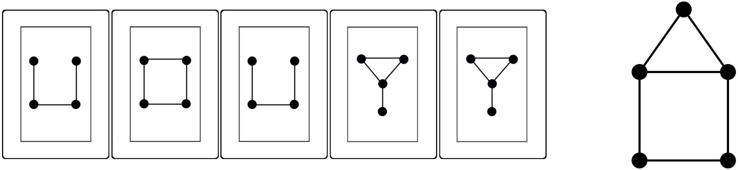

Plausibility — types of results

Classes of reconstructible graphs:
Trees, disconnected graphs, maximal planar graphs, ...

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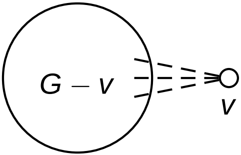
Trees, disconnected graphs, maximal planar graphs, ...

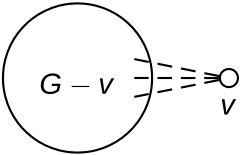
What we can reconstruct *about any* graph:
Number of vertices and edges, characteristic polynomial, number of copies of a fixed subgraph, ...

Example



Card	**Number of edges**		
$G - v_1$	$e(G - v_1)$	$=$	$e(G) - d_G(v_1)$
$\vdots$		$\vdots$	
$G - v_n$	$e(G - v_n)$	$=$	$e(G) - d_G(v_n)$
	$\sum e(G - v_i)$	$=$	$(n-2)e(G)$

Example



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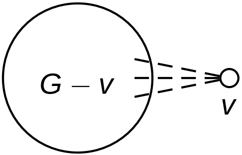
We can reconstruct **the degree sequence**.

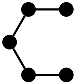
Example



Card	**Number of edges**	
$G - v_1$	$e(G - v_1) = e(G) - d_G(v_1)$	$\therefore d_G(v_1) = \dots$
$\vdots$	$\vdots$	$\vdots$
$G - v_n$	$e(G - v_n) = e(G) - d_G(v_n)$	$\therefore d_G(v_n) = \dots$
	$\sum e(G - v_i) = (n-2)e(G)$	

We can reconstruct **the degree sequence**.

We can reconstruct **regular graphs**.



card

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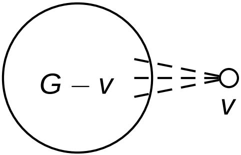

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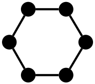
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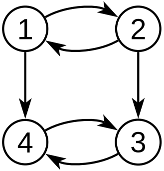
Digraph reconstruction

Conjecture (Harary)

For large enough n, no two nonisomorphic n-vertex digraphs have the same deck.



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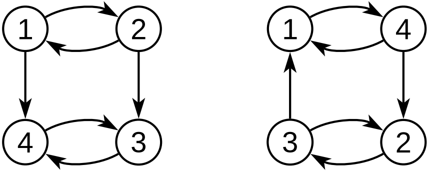

Digraph reconstruction

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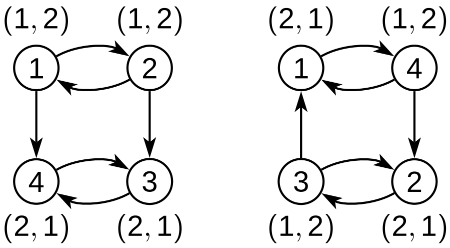
Theorem (Manvel, 1973)

The (indegree, outdegree)-pair sequence of any digraph on ≥ 5 *vertices can be determined from the deck.*



Bad news

Theorem (Stockmeyer, 1977, 1981, 1988)

There are infinitely many pairs of non-reconstructible digraphs.

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There are infinitely many pairs of non-reconstructible digraphs.

Other non-reconstructible objects:

- Graph orientiations, eg. tournaments
- Hypergraphs (even 3-uniform hypergraphs)
- Infinite graphs (even locally-finite countable forests)

Still open

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Similar vs. pseudosimilar vertices

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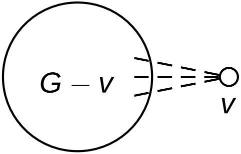
Similar vs. pseudosimilar vertices

Differences between graphs and digraphs



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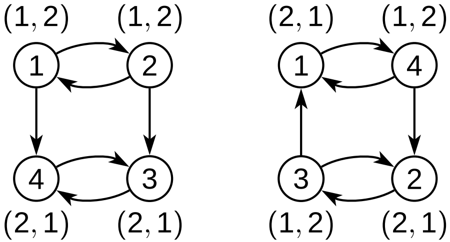
We can determine the degree sequence of the graph **and which degree goes with which deleted vertex.**

Differences between graphs and digraphs

Theorem (Manvel, 1973)

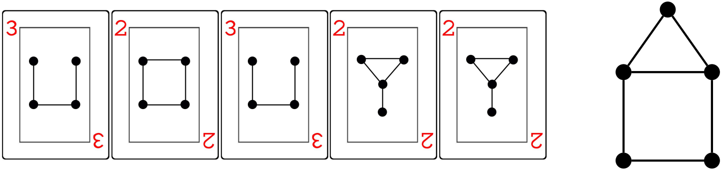
The (indegree, outdegree)-pair sequence of any digraph on ≥ 5 vertices can be determined from the deck.



We can determine the degree pairs, **but not the vertices to which they belong.**

Degree-associated reconstruction

Each card is presented with the degree of the deleted vertex.



degree-associated cards, deck (dacards, dadeck)

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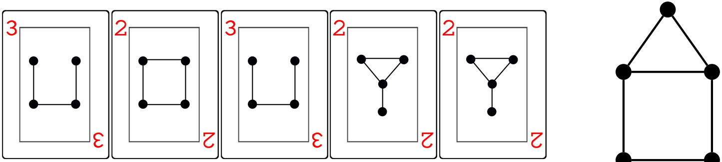

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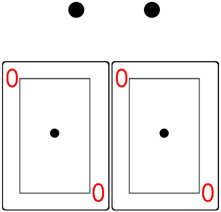

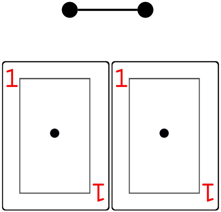

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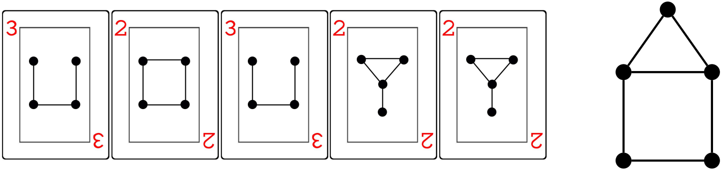
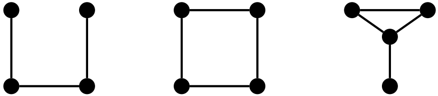


degree-associated cards, deck (dacards, dadeck)

Conjecture (Ramachandran, 1979)

No two nonisomorphic graphs or digraphs (on any number of vertices) have the same dadeck.

Operating without a full deck



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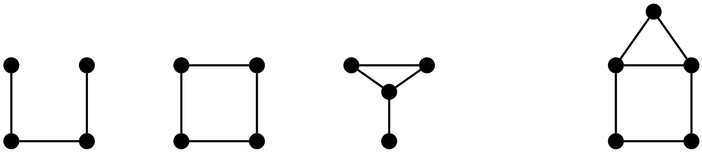

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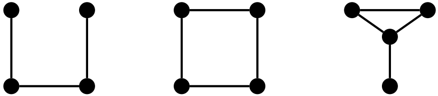

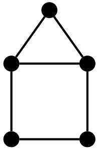

Reconstruction number rn(G): Size of a smallest “subdeck” that uniquely determines G.
(Harary–Plantholt, 1985)

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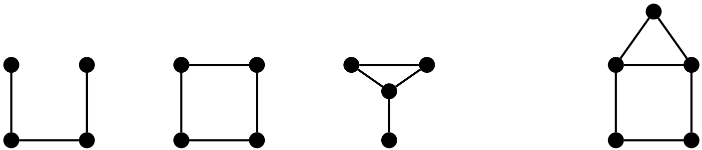

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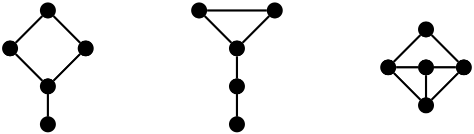

The degree-associated version (B, West 2010)

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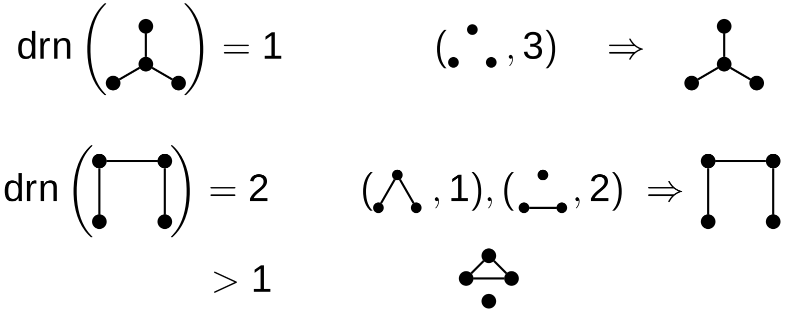
Degree-associated reconstruction number drn(G): Size of a smallest subdadeck that uniquely determines G.
(Ramachandran, 2006)



Typical reconstruction number values

Observation

For all G, $\mathrm{drn}(G) \leq \mathrm{rn}(G)$

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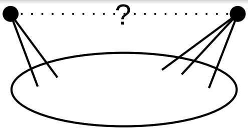
Theorem (Bollobás, 1990)

Almost every graph is uniquely determined by 3 cards. Furthermore, for almost every graph, any two cards in the deck determine everything about the graph except whether there is an edge joining the two deleted vertices.



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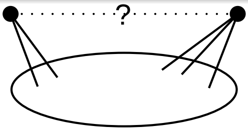

Corollary

- *For almost every graph* G, $\mathrm{rn}(G) = 3$.

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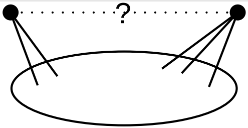
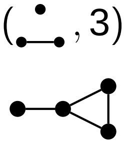
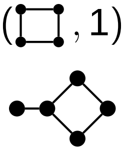


Corollary

- *For almost every graph G,* $\mathrm{rn}(G) = 3$.
- *Almost every graph G satisfies* $\mathrm{drn}(G) \leq 2$.

The smallest drn possible





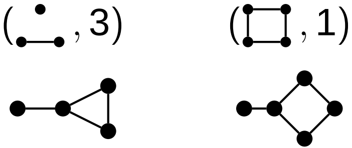
The smallest drn possible

Theorem

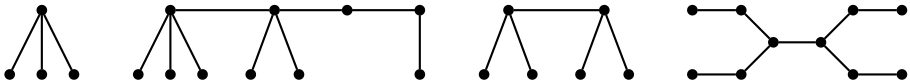
drn$(G) = 1$ *if and only if one of the the following holds:*

(1) *G has a vertex of degree d, where $d \in \{0, |V(G)| - 1\}$;*

(2) *G has a vertex v of degree d, where $d \in \{1, |V(G)| - 2\}$, such that $G - v$ is a vertex-transitive graph.*



Trees



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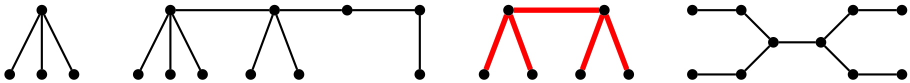

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With the exception of stars and the graph highlighted above, every caterpillar C *satisfies* $\mathrm{drn}(C) = 2$.

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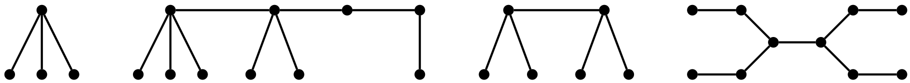

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For any tree T *on at least 5 vertices, there exist 3 cards that uniquely determine* T.

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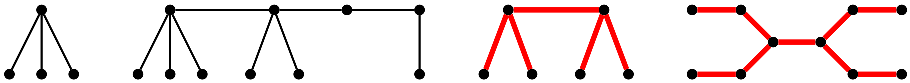

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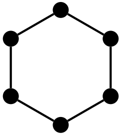
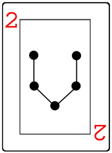
Conjecture

For all but finitely many trees T, $\text{drn}(T) \leq 2$.

Larger drn values: vertex-transitive graphs





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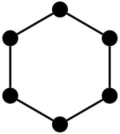

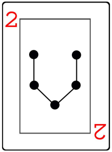

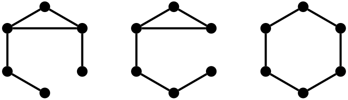

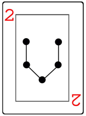

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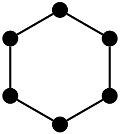

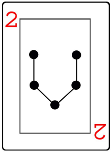

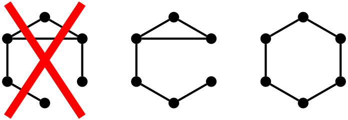

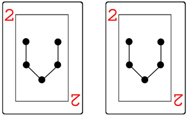

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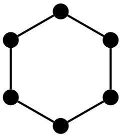

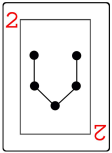

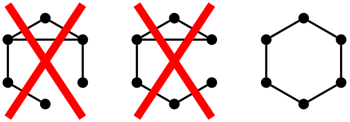

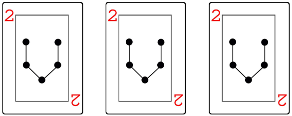

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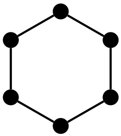

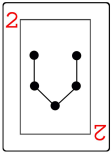

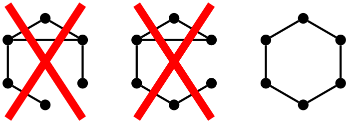

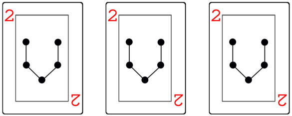

Proposition

If G is vertex-transitive but not complete or edgeless, then $\mathrm{drn}(G) \geq 3$.

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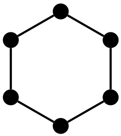

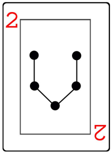

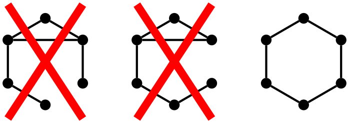

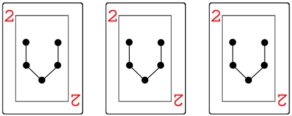

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Proposition

If G is r-regular, then $\mathrm{drn}(G) \leq \min\{r+2, n-r+1\}$.

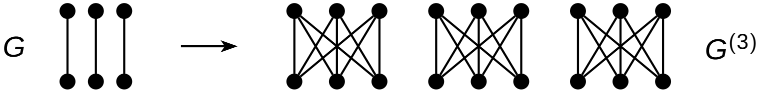
Vertex-transitive graphs

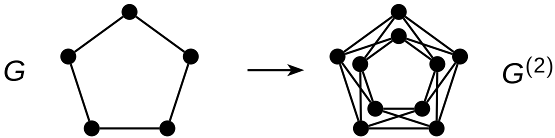
Theorem (Ramachandran, 2000, 2006)

- $\text{drn}(K_n) = 1$ *for all* $n \geq 1$
- $\text{drn}(C_n) = 3$ *for* $n \geq 4$
- $\text{drn}(K_{m,m}) = 3$ *for* $m \geq 2$
- $\text{drn}(tK_n) = 3$ *for* $t, n \geq 2$
- $\text{drn}(tK_{m,m}) = m + 2$ *for* $t, m \geq 2$

A construction

For non-complete, vertex-transitive G without “twins,” replace each vertex by m pairwise nonadjacent vertices, where $m \geq 2$, and replace each edge of the original graph with a copy of $K_{m,m}$.





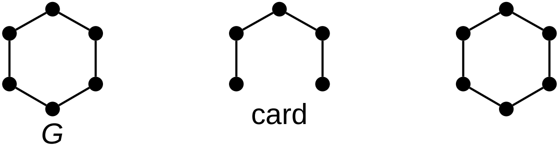
Theorem

With G as above, $\mathrm{drn}(G^{(m)}) = m + 2$.

Coherence

A vertex-transitive graph *G* is **coherent** if whenever we delete two vertices from it, the only way to add a vertex back to get a card of *G* is to “put one of the missing vertices back.”

Not coherent:



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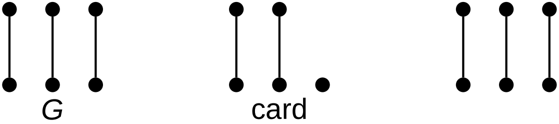

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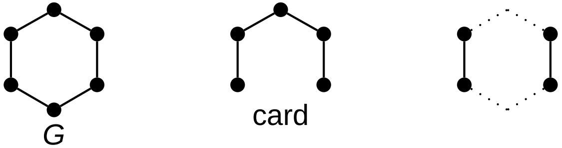

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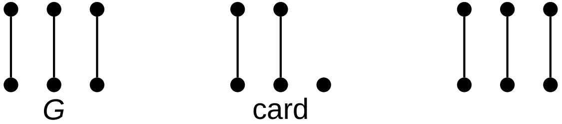

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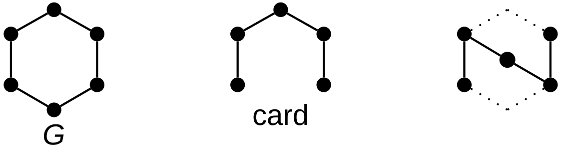

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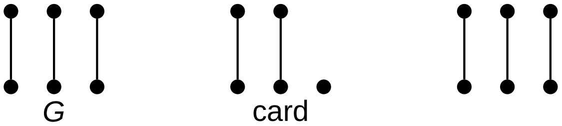

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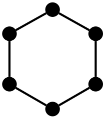

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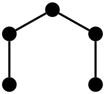

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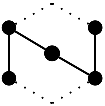

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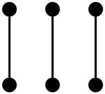

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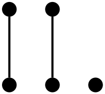

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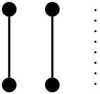

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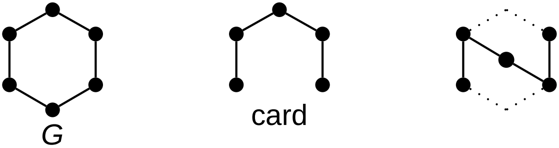

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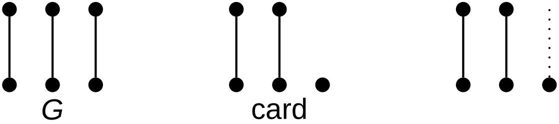

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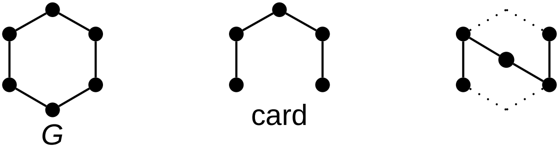

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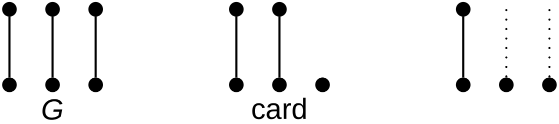

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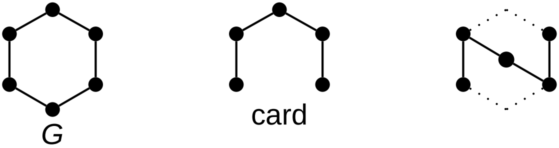

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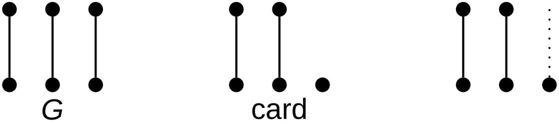

Coherence and drn 3

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If G is a coherent vertex-transitive graph such that G has no twins, then $\mathrm{drn}(G) = 3$.

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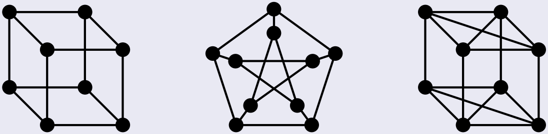
Theorem

The following graphs are all coherent and have $\mathrm{drn} = 3$*.*

- *the* d*-dimensional hypercube* Q_d*;*
- *the Petersen graph;*
- $K_n \square K_2$.



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- Are "most" vertex-transitive graphs coherent or not? Do "most" have drn equal to 3?
- Are there infinitely many (vertex-transitive) graphs G with $\text{drn}(G) \geq \alpha n(G)$ for $0 < \alpha \leq 1$?

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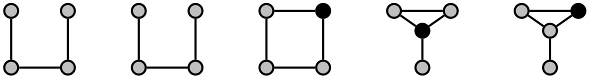

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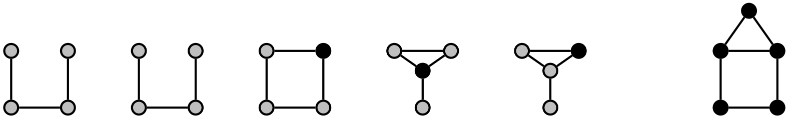

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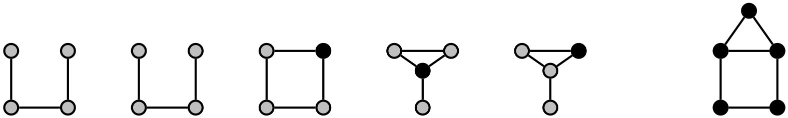

Conjecture (Stockmeyer, 1988)

The Reconstruction Conjecture is not true, but the smallest counterexample is on 87 vertices and will never be found.