Perfect graphs I: Origins, a theorem, and a conjecture

Michael D. Barrus

Department of Mathematics
Brigham Young University

Graphs and Matrices Seminar
October 3, 2012

Definitions

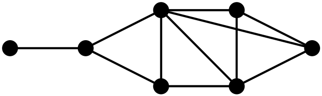
χ: chromatic number	ω: clique number	$\chi \geq \omega$
θ: clique cover number	α: independence number	$\theta \geq \alpha$

χ-perfect: Every induced subgraph satisfies $\chi = \omega$.
α-perfect: Every induced subgraph satisfies $\alpha = \theta$.

CLASS 1: Graphs for which $\Theta = \alpha$ for every induced subgraph.
CLASS 2: Graphs for which $\alpha = \theta$ for every induced subgraph.
CLASS 3: Graphs for which $\chi = \omega$ for every induced subgraph.
CLASS 4: Graphs containing no induced odd cycles of length $\geq$ 5 or their complements.

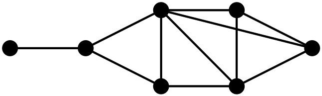
Chordal graphs

Graphs where every cycle of length at least 4 has a chord (i.e., graphs with no induced cycles of length at least 4).



Chordal graphs

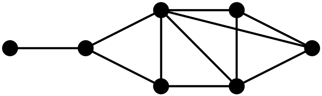
Graphs where every cycle of length at least 4 has a chord (i.e., graphs with no induced cycles of length at least 4).



Chordal graphs are α-perfect [Hajnal, Suranyi, 1959].

Chordal graphs

Graphs where every cycle of length at least 4 has a chord (i.e., graphs with no induced cycles of length at least 4).

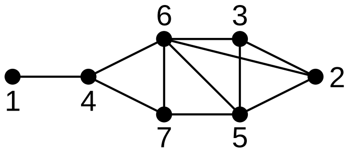


Chordal graphs are α-perfect [Hajnal, Suranyi, 1959].

Chordal graphs are χ-perfect [Berge, 1960].

Chordal graphs

Graphs where every cycle of length at least 4 has a chord (i.e., graphs with no induced cycles of length at least 4).



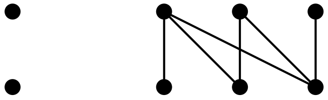
Chordal graphs are α-perfect [Hajnal, Suranyi, 1959].

Chordal graphs are χ-perfect [Berge, 1960].

Chordal graphs have **simplicial orderings**.

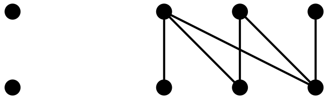
Bipartite graphs

Graphs with $\chi \leq 2$. (Equivalently, graphs with no odd cycles.)



Bipartite graphs

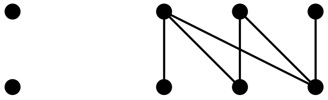
Graphs with $\chi \leq 2$. (Equivalently, graphs with no odd cycles.)



Bipartite graphs are χ-perfect.

Bipartite graphs

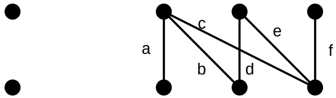
Graphs with $\chi \leq 2$. (Equivalently, graphs with no odd cycles.)



Bipartite graphs are χ-perfect. They are also α-perfect [König, 1916].

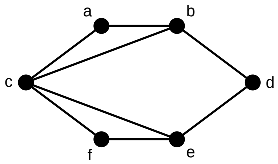
Bipartite graphs

Graphs with $\chi \leq 2$. (Equivalently, graphs with no odd cycles.)



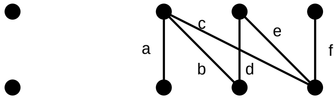
Bipartite graphs are χ-perfect. They are also α-perfect [König, 1916].

Line graphs of bipartite graphs



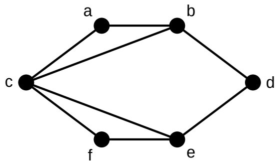
Bipartite graphs

Graphs with $\chi \leq 2$. (Equivalently, graphs with no odd cycles.)



Bipartite graphs are χ-perfect. They are also α-perfect [König, 1916].

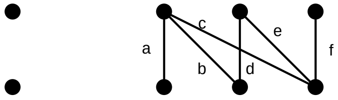
Line graphs of bipartite graphs



Line graphs of bipartite graphs are α-perfect [König, Egerváry, 1931].

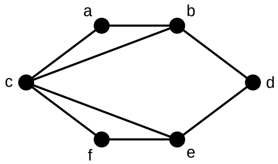
Bipartite graphs

Graphs with $\chi \leq 2$. (Equivalently, graphs with no odd cycles.)



Bipartite graphs are χ-perfect. They are also α-perfect [König, 1916].

Line graphs of bipartite graphs

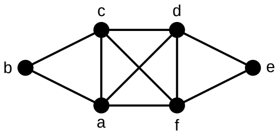


Line graphs of bipartite graphs are α-perfect [König, Egerváry, 1931].

They are also χ-perfect [König, 1916].

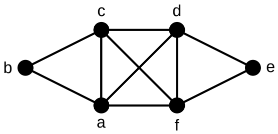
Comparability graphs

Graphs modeling relationships in a poset.



Comparability graphs

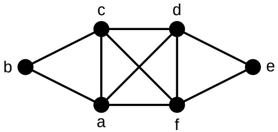
Graphs modeling relationships in a poset.



Comparability graphs are χ-perfect [Mirsky, 1971].

Comparability graphs

Graphs modeling relationships in a poset.



Comparability graphs are χ-perfect [Mirsky, 1971].

Comparability graphs are α-perfect [Dilworth, 1950].

Berge's conjectures (early 1960's)

The Weak Perfect Graph Conjecture

Class 2 = Class 3.

In other words, a graph is χ-perfect [α-perfect] if and only if its complement is.

The Strong Perfect Graph Conjecture

Classes 2 and 3 are the same as Class 4.

*In other words, the χ-perfect [α-perfect] graphs are exactly those graphs having no induced **odd hole** or **odd antihole**.*

Berge also conjectured that Class 4 $\subseteq$ Class 1.

The (Weak) Perfect Graph Theorem

Theorem (Lovász, 1972)

A graph is perfect if and only if $\omega(H)\alpha(H) \geq |V(H)|$ *for every induced subgraph* H.

Corollary (Perfect Graph Theorem)

A graph G *is perfect if and only if* $\overline{G}$ *is perfect.*

Proof sketch (Gasparyan, 1996)

Note that $\omega(H)\alpha(H) \geq |V(H)|$ for induced subgraphs of χ-perfect graphs.

Proof sketch (Gasparyan, 1996)

Note that $\omega(H)\alpha(H) \geq |V(H)|$ for induced subgraphs of χ-perfect graphs.

Approach: Show that $\omega(H)\alpha(H) < |V(H)|$ for some H in every imperfect graph.

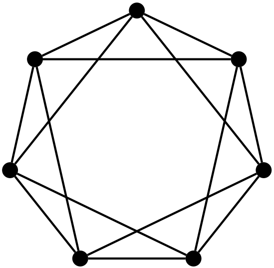
Proof sketch (Gasparyan, 1996)

Note that $\omega(H)\alpha(H) \geq |V(H)|$ for induced subgraphs of χ-perfect graphs.

Approach: Show that $\omega(H)\alpha(H) < |V(H)|$ for some H in every imperfect graph.

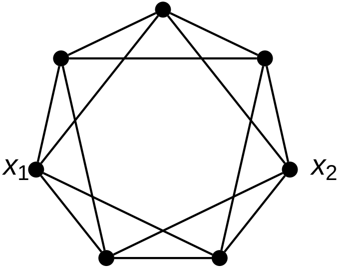
It suffices to consider **p-critical** subgraphs (minimal imperfect subgraphs.)

$\overline{C_7}$



Proof sketch, continued

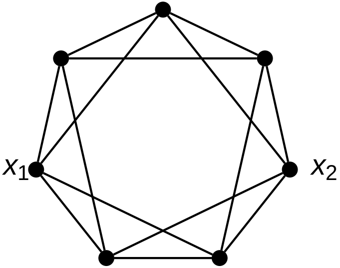
$a = 2, w = 3$



Let $a = \alpha(G)$ and $w = \omega(G)$. Let $S_0 = \{x_1, \ldots, x_a\}$ be a maximum independent set in G.

Proof sketch, continued

$a = 2, w = 3$



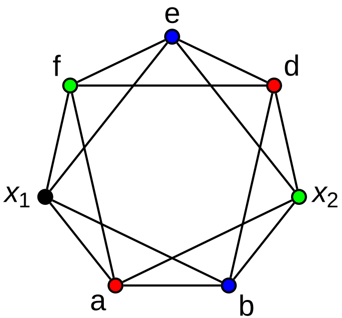
Let $a = \alpha(G)$ and $w = \omega(G)$. Let $S_0 = \{x_1, \ldots, x_a\}$ be a maximum independent set in G.

Lemma: Deleting any independent set in a p-critical graph leaves the clique number unchanged.

Hence $G - x_r$ is w-colorable, and the color classes partition the rest of the vertices into sets S_i

Proof sketch, continued

$a = 2, w = 3$



Let $a = \alpha(G)$ and $w = \omega(G)$. Let $S_0 = \{x_1, \ldots, x_a\}$ be a maximum independent set in G.

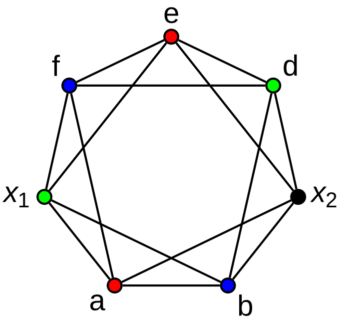
Lemma: Deleting any independent set in a p-critical graph leaves the clique number unchanged.

Hence $G - x_r$ is w-colorable, and the color classes partition the rest of the vertices into sets S_i

$S_0 = \{x_1, x_2\}$, $S_1 = \{$a, d$\}$, $S_2 = \{$b,e$\}$, $S_3 = \{x_2,$ f$\}$

Proof sketch, continued

$a = 2, w = 3$



Let $a = \alpha(G)$ and $w = \omega(G)$. Let $S_0 = \{x_1, \ldots, x_a\}$ be a maximum independent set in G.

Lemma: Deleting any independent set in a p-critical graph leaves the clique number unchanged.

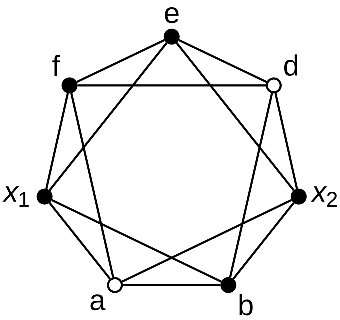
Hence $G - x_r$ is w-colorable, and the color classes partition the rest of the vertices into sets S_i

$S_0 = \{x_1, x_2\}$, $S_1 = \{$a, d$\}$, $S_2 = \{$b,e$\}$, $S_3 = \{x_2, f\}$,
$S_4 = \{$a, e$\}$, $S_5 = \{$b,f$\}$, $S_6 = \{x_1, d\}$

$aw + 1$ (overlapping) sets in all

Proof sketch, continued

$S_1 = \{a, d\}$, $Q_1 = \{x_1, \text{e}, \text{f}\}$



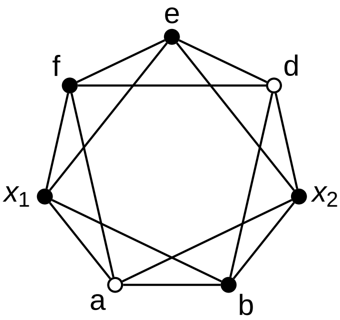
Each S_i is an independent set.

Lemma: Deleting any independent set in a p-critical graph leaves the clique number unchanged.

Hence $G - S_i$ contains a maximum clique Q_i that is maximum in G.

Proof sketch, continued

$S_1 = \{a, d\}$, $Q_1 = \{x_1, \text{e, f}\}$



Each S_i is an independent set.

Lemma: Deleting any independent set in a p-critical graph leaves the clique number unchanged.

Hence $G - S_i$ contains a maximum clique Q_i that is maximum in G.

Lemma: Each Q_i intersects S_j when $i \neq j$.

Proof sketch, concluded

Let A and B be $|V(G)|$-by-$(aw+1)$ incidence matrices for the families S_i and Q_i.

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 \end{bmatrix}$$

Proof sketch, concluded

Let A and B be $|V(G)|$-by-$(aw+1)$ incidence matrices for the families S_i and Q_i.

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 \end{bmatrix}$$

Since $|S_i \cap Q_j| = 1$ when $i \neq j$, and $|S_i \cap Q_i| = 0$, we have

$$A^T B = J - I.$$

Proof sketch, concluded

Let A and B be $|V(G)|$-by-$(aw+1)$ incidence matrices for the families S_i and Q_i.

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 \end{bmatrix}$$

Since $|S_i \cap Q_j| = 1$ when $i \neq j$, and $|S_i \cap Q_i| = 0$, we have

$$A^T B = J - I.$$

Rank $J - I$ = $aw + 1$, so A and B have rank at least $aw + 1$. Thus $|V(G)| \geq aw + 1$ and hence $aw < |V(G)|$. $\square$

Berge's conjectures (early 1960's)

The (Weak) Perfect Graph Theorem

Class 2 = Class 3.

In other words, a graph is χ-perfect [α-perfect] if and only if its complement is.

The Strong Perfect Graph Conjecture

Classes 2 and 3 are the same as Class 4.

In other words, the χ-perfect [α-perfect] graphs are exactly those graphs having no induced ***odd hole*** *or* ***odd antihole****.*