The Erdős-Gallai differences of a degree sequence

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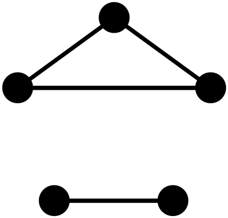
New York Combinatorics Seminar
October 8, 2021

Definitions

The **degree sequence** $d = (d_1, \ldots, d_n)$ of a simple graph records the number of edges incident with each vertex. We write it in nonincreasing order.



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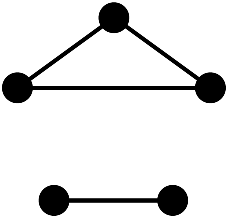

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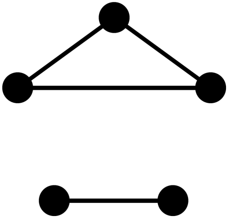

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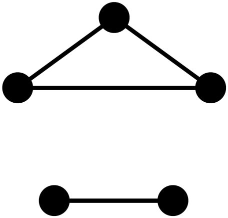

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$$\boxed{\Delta_k(d) = k(k-1) + \sum_{i>k} \min\{k, d_i\} - \sum_{i \le k} d_i} \qquad \text{for } 1 \le k \le m(d),$$

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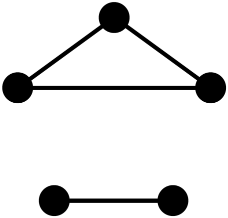

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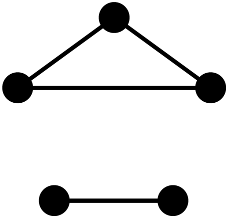

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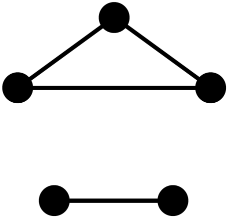

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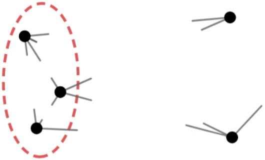
Where the definition comes from

Theorem [Erdős–Gallai '60; Li '75; Hammer–Ibaraki–Simeone '81]

A nonincreasing list of nonnegative numbers $(d_1, \ldots, d_n)$ with even sum **is the degree sequence of a simple graph** iff

$$\sum_{i \leq k} d_i \leq k(k-1) + \sum_{i > k} \min\{k, d_i\}$$

for all $k \in \{1, \ldots, m(d)\}$.



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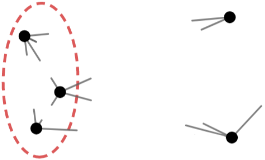

Corollary

For any degree sequence d and $k \in \{1, \ldots, m(d)\}$,

$$\Delta_k(d) = k(k-1) + \sum_{i>k} \min\{k, d_i\} - \sum_{i \le k} d_i \quad \ge \quad 0.$$

Motivation/applications

First appearances

Erdős–Gallai differences appear in Koren '75 and (with an opposite sign) Li '75, used in **simplifying degree sequence recognition** and **characterizing special graph classes**.

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The Erdős–Gallai criterion: $\Delta_k(d) \geq 0$ for all k

Lemma [Li '75]

Given a degree sequence d with n terms, let $m = m(d)$. Then

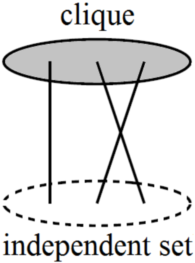
- The differences $\Delta_m(d), \Delta_{m+1}(d), \ldots, \Delta_n(d)$ form a strictly increasing sequence.
- If $d_1 = \cdots = d_q > d_{q+1}$ and $\Delta_q(d)$ is nonnegative, then $\Delta_i(d) \geq 0$ for all $i \in \{1, \ldots, \min\{q, m\}\}$.

Later authors: even fewer Erdős–Gallai inequalities need to be checked!

Split graphs and the last EG difference

Erdős–Gallai differences show up in the **splittance** $s(G)$ of a graph (i.e., the **edit distance** to the class of split graphs).

Split graphs



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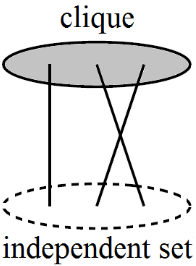

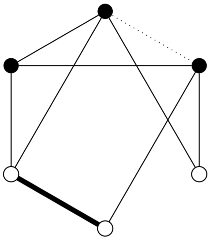

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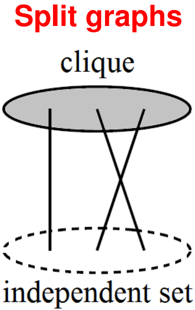

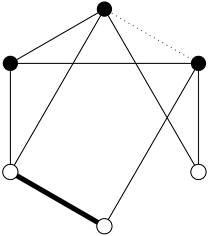

$$\Delta(d) = (2, 4, 4, 4)$$

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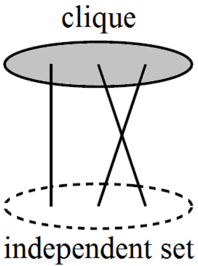
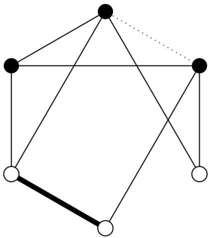




$\Delta(d) = (2, 4, 4, 4)$

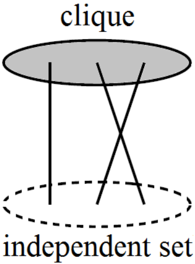
Theorem [Hammer–Simeone '81], adapted

If d is the degree sequence of any graph G, then $\Delta_{m(d)}(d) = 2s(G)$.

Graph families

Other degree sequence characterizations can be restated in terms of Erdős–Gallai differences.

Split graphs



EXAMPLE. $\Delta(d) = (2, 1, 0)$

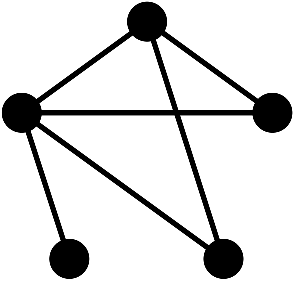
Theorem [Hammer–Simeone '81], adapted

A graph with degree sequence d is a split graph iff $\Delta_{m(d)}(d) = 0$.

Graph families

Other degree sequence characterizations can be restated in terms of Erdős–Gallai differences.

Threshold graphs



EXAMPLE. $\Delta(d) = (0, 0, 0)$

Threshold: uniquely realizable from its degree sequence; also...

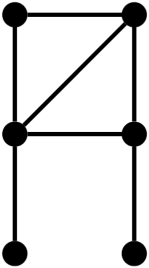
Theorem [Li '75, Hammer–Ibaraki–Simeone '81], adapted

A graph with degree sequence d is a threshold graph iff $\Delta_k(d) = 0$ for all $k \in \{1, \ldots, m(d)\}$.

Graph families

Other degree sequence characterizations can be restated in terms of Erdős–Gallai differences.

Weakly threshold graphs



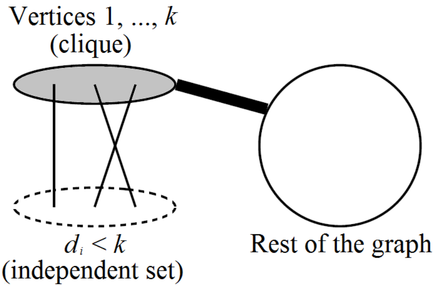
EXAMPLE. $\Delta(d) = (1, 1, 0)$

Definition [B '18]

A graph with degree sequence d is a weakly threshold graph iff $\Delta_k(d) \leq 1$ for all $k \in \{1, \ldots, m(d)\}$.

Further motivation/applications

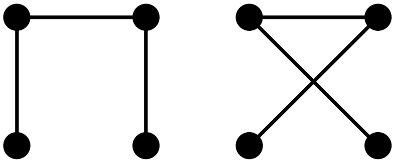
Decomposition
(Koren '75, Tyshkevich '00, B '13)



When $\Delta_k(d) = 0$

Forced adjacencies
(B '18, Cloteaux '19)

$d = (2, 2, 1, 1)$



$\Delta(d) = (1, 0)$

When $\Delta_k(d) \leq 1$

What more can (principal) EG differences tell us?

Principal Erdős–Gallai differences

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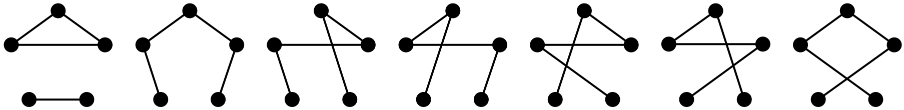

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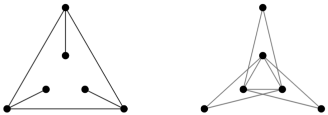
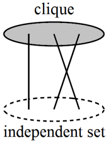
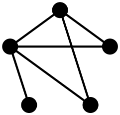
A common theme: complements

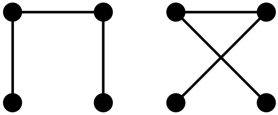


Proposition [Földes–Hammer ’77, Chvátal–Hammer ’77, B ’18]

A graph { is split / is threshold / is weakly threshold / is decomposable / has forced (non-)adjacencies } iff its complement is/does.







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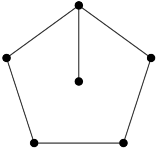
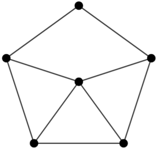


$(2, 4, 4)$



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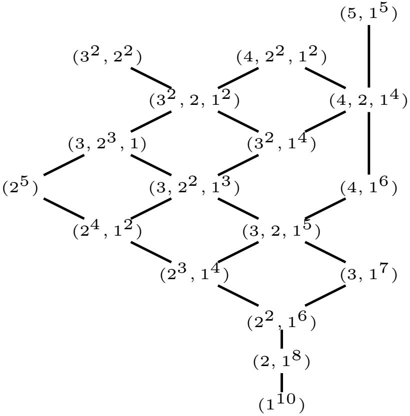
A common theme: majorization

$$d \succeq e \quad \text{if} \quad \sum_{i \le k} d_i \ge \sum_{i \le k} e_i \text{ for all } k$$



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Degree sequences for the following classes are "upwards closed" in the poset:

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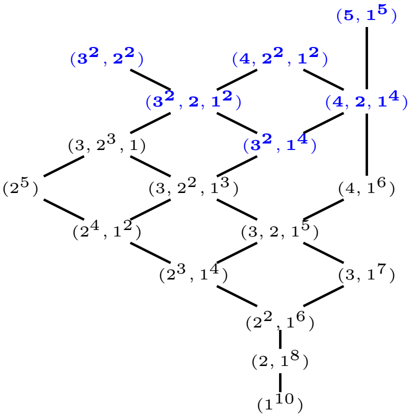
- [Merris, 2003] Split graphs
- [B, 2018] Weakly threshold graphs, decomposable graphs, graphs with forced (non-)adjacencies



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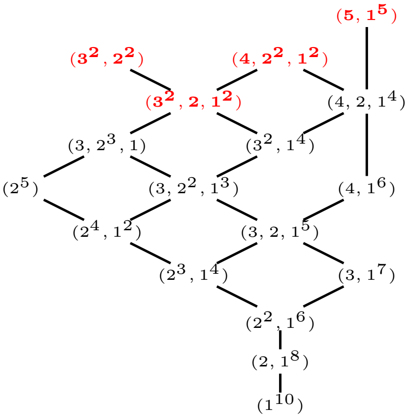
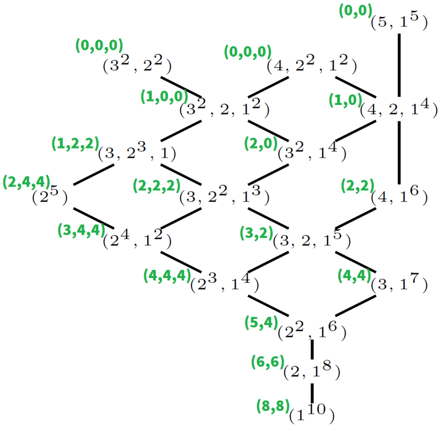


Today's questions, part 2

What characteristics of the Erdős–Gallai differences are preserved as one moves upward/downward through a poset of degree sequences?



A useful tool

The **corrected Ferrers diagram** $F(d)$ of a degree sequence $d = (d_1, \ldots, d_n)$ is a $(0,1)$-matrix with 0's along the diagonal and, otherwise, d_i left-justified 1's in the ith row.

$F((3,2,2,2,2,1))$

0	1	1	1	0	0
1	0	1	0	0	0
1	1	0	0	0	0
1	1	0	0	0	0
1	1	0	0	0	0
1	0	0	0	0	0

$F(\overline{d})$

0	1	1	1	1	0
1	0	1	1	0	0
1	1	0	1	0	0
1	1	1	0	0	0
1	1	1	0	0	0
1	1	0	0	0	0

Rotated, toggled version of the original!

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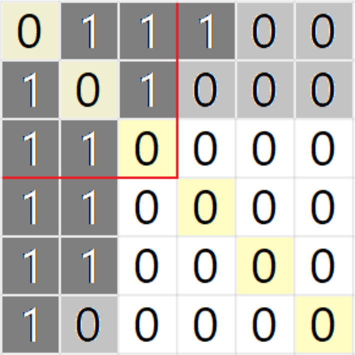


$$\sum_{i \leq k} d_i$$

$$k(k-1) + \sum_{i>k} \min\{k, d_i\}$$

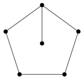
A helpful reformulation

Define the (new?) **difference matrix** $M(d)$ by

$$M(d) = F(d)^T - F(d).$$

$$M((3,2,2,2,2,1)) = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$d = (3,2,2,2,2,1)$
$\Delta(d) = (2,4,4)$



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Observation [B, '21+]

For $k \in \{1, \ldots, m(d)\}$, the sum of the entries in the first k rows of $M(d)$ equals $\Delta_k(d)$.

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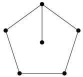

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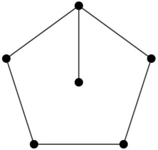

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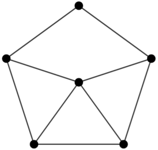

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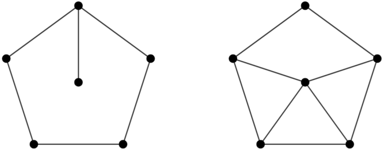


$(2, 4, 4)$ $(1, 3, 4, 4)$

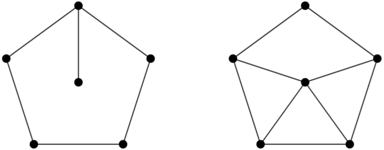
Is this immediate?

If nonincreasing $d = (d_1, \ldots, d_n)$ is a degree sequence, then the complement of a realization has degree sequence $\overline{d} = (n - 1 - d_n, \ldots, n - 1 - d_1)$:

$$\Delta_k(\overline{d}) = k(k-1) + \sum_{i>k} \min\{k, n-1-d_{n+1-i}\} - \sum_{i \le k} (n-1-d_{n+1-i}).$$

Today's questions, part 1

Can we "connect" the Erdős–Gallai differences of complementary graphs somehow?



$(2, 4, 4)$ $(1, 3, 4, 4)$

Is this immediate?

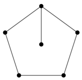
If nonincreasing $d = (d_1, \ldots, d_n)$ is a degree sequence, then the complement of a realization has degree sequence $\overline{d} = (n-1-d_n, \ldots, n-1-d_1)$:

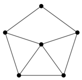
$$\Delta_k(\overline{d}) = k(k-1) + \sum_{i>k} \min\{k, n-1-d_{n+1-i}\} - \sum_{i \le k} (n-1-d_{n+1-i}).$$

Maybe not?

Comparing complements

$d = (3,2,2,2,2,1)$
$\Delta(d) = (2,4,4)$





$\overline{d} = (4,3,3,3,3,2)$
$\Delta(\overline{d}) = (1,3,4,4)$

$$M(d) = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} 0 \\ \color{red}{2} \\ \color{purple}{4} \\ \color{purple}{4} \\ \color{blue}{3} \\ \color{blue}{1} \\ 0 \end{matrix}$$

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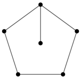

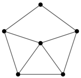

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Theorem [B, '21+]

For all degree sequences d,

- $M(\overline{d}) = M(d)_{\perp}$, where $A_{\perp}$ denotes the "transpose about the antidiagonal."
- The later sums of entries by rows give the terms of $\Delta(\overline{d})$ in reverse order.

Proof

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$F(\overline{d})$ is obtained by rotating $F(d)$ by 180° and switching non-diagonal entries from 0 to 1, and vice versa:

$$F(\overline{d}) = (J_n - I_n - F(d))_\perp^T,$$

where I_n is the identity and J_n is the all-ones matrix.

0	1	1	1	0	0
1	0	1	0	0	0
1	1	0	0	0	0
1	1	0	0	0	0
1	1	0	0	0	0
1	0	0	0	0	0

0	1	1	1	1	0
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where I_n is the identity and J_n is the all-ones matrix. Then

$$\begin{aligned} M(\overline{d}) &= F(\overline{d})^T - F(\overline{d}) \\ &= [(J_n - I_n - F(d))_\perp^T]^T - (J_n - I_n - F(d))_\perp^T \\ &= \Big(F(d)^T - F(d)\Big)_\perp \\ &= M(d)_\perp. \end{aligned}$$

Comparing complements

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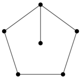

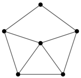

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Theorem [B, '21+]

For all degree sequences d,

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Proof, continued

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For all degree sequences d,

- $M(\overline{d}) = M(d)_\perp$, where $A_\perp$ denotes the "transpose about the antidiagonal."
- The later sums of entries by rows give the terms of $\Delta(\overline{d})$ in reverse order.

Letting

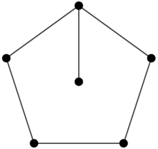
$$h_k = [\underbrace{1 \quad \cdots \quad 1}_{k \text{ terms}} \underbrace{0 \quad \cdots \quad 0}_{n-k \text{ terms}}], \qquad \mathbf{1}^T = [1 \quad \cdots \quad 1],$$

consider the sum of entries in the first k rows of $M(\overline{d})$:

$$\begin{aligned} \Delta_k(\overline{d}) &= h_k M(\overline{d})\mathbf{1} & &= (\mathbf{1}^T - h_{n-k})M(d)^T\mathbf{1} \\ &= h_k M(d)_\perp \mathbf{1} & &= \mathbf{1}^T M(d)^T \mathbf{1} - h_{n-k}M(d)^T\mathbf{1} \\ &= (\mathbf{1}^T - h_{n-k})_\perp^T M(d)_\perp \mathbf{1}_\perp^T & &= 0 - h_{n-k}M(d)^T\mathbf{1} \\ &= [(\mathbf{1}^T - h_{n-k})M(d)^T\mathbf{1}]_\perp^T & &= h_{n-k}M(d)\mathbf{1}. \end{aligned}$$

Today's question

Can we "connect" the Erdős–Gallai differences of complementary graphs somehow?



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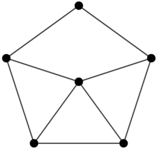

$(1,3,4,4)$

$M(d) =$

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} 0 \\ 2 \\ 4 \\ 4 \\ 3 \\ 1 \\ 0 \end{matrix}$$

Is there a connection between cumulative sums of **the first rows** and that of **the all-but-last-few rows**?

Collisions and islands in $M(d), M(\overline{d})$

$$M(d) = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} 0 \\ 2 \\ 4 \\ 4 \\ 3 \\ 1 \\ 0 \end{matrix}$$

Note how the sequences of differences overlap at their ends...

$$M(\overline{d}) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & -1 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} 0 \\ 1 \\ 3 \\ 4 \\ 4 \\ 2 \\ 0 \end{matrix}$$

An “island” of nonzero entries has the same sum when “anti-transposed”...

An answer to the question

Theorem [B, '21+]

Some (though perhaps not all) of the Erdős–Gallai differences of $\overline{d}$ match those of d.

In particular,

- $\Delta(d)$ and $\Delta(\overline{d})$ have the same **final** term, and
- $\Delta(d)$ and $\Delta(\overline{d})$ have the same **maximum** term.

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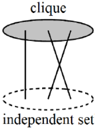
Split graphs

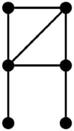


EXAMPLE. $\Delta(d) = (2, 1, 0)$

Theorem [Hammer–Simeone ’81], adapted

A graph with degree sequence d is a split graph iff $\Delta_{m(d)}(d) = 0$.

Weakly threshold graphs



EXAMPLE. $\Delta(d) = (1, 1, 0)$

Definition [B ’18]

A graph with degree sequence d is a weakly threshold graph iff $\Delta_k(d) \leq 1$ for all $k \in \{1, \ldots, m(d)\}$.

An answer to the question

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Some (though perhaps not all) of the Erdős–Gallai differences of $\overline{d}$ match those of d.

In particular,

- $\Delta(d)$ and $\Delta(\overline{d})$ have the same **final** term, and
- $\Delta(d)$ and $\Delta(\overline{d})$ have the same **maximum** term.

Corollary

$$\text{A graph is } \left\{ \begin{array}{c} \text{split} \\ \text{threshold} \\ \text{weakly threshold} \end{array} \right\} \text{ iff its complement is.}$$

More generally

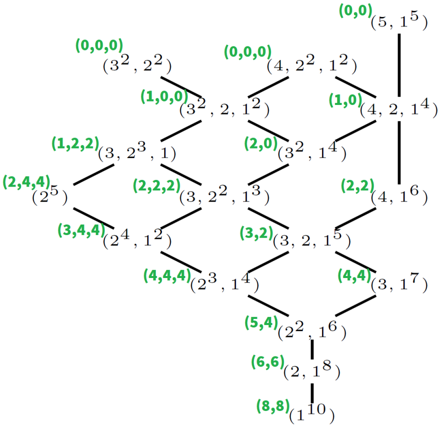
Corollary [B '21+]

The following classes of graphs are closed under complementation for all $z \in \mathbb{N}_0$:

- Those graphs having final Erdős–Gallai difference equal to (or $\leq$) z (i.e., those with splittance (at most) $z/2$);
- Those graphs having maximum Erdős–Gallai difference equal to (or $\leq$) z.

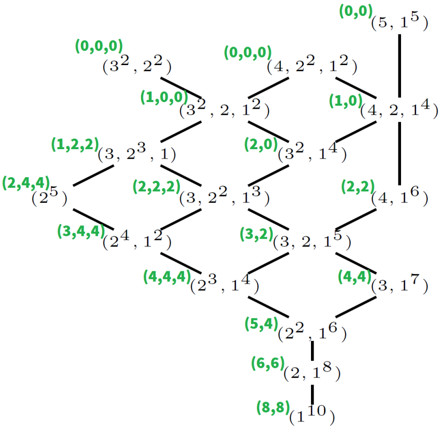
Today's questions, part 2

What characteristics of the Erdős–Gallai differences are preserved as one moves upward/downward through a poset of degree sequences?



In the majorization poset

$$d \succeq e \quad \text{if} \quad \sum_{i \le k} d_i \ge \sum_{i \le k} e_i \ \text{ for all } k$$



Theorem [B, 21+]

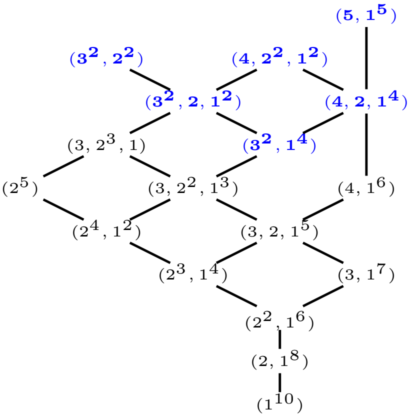
- If $d \succeq e$, then $\Delta_k(d) \le \Delta_k(e)$ for all $k \in \{1, \ldots, m'\}$, where $m' = \min\{m(d), m(e)\}$.
- If $d \succeq e$, then $\Delta_{m(d)}(d) \le \Delta_{m(e)}(e)$.
- If $d \succeq e$, then $\max \Delta_j(d) \le \max \Delta_k(e)$.

Monotonicity under majorization

$$d \succeq e \quad \text{if} \quad \sum_{i \le k} d_i \ge \sum_{i \le k} e_i \text{ for all } k$$

Degree sequences for the following classes are "upwards closed" in the poset:

- [Ruch–Gutman, 1979; Peled–Srinivasan, 1989] Threshold graphs
- [Merris, 2003] Split graphs
- [B, 2018] Weakly threshold graphs, decomposable graphs, graphs with forced (non-)adjacencies



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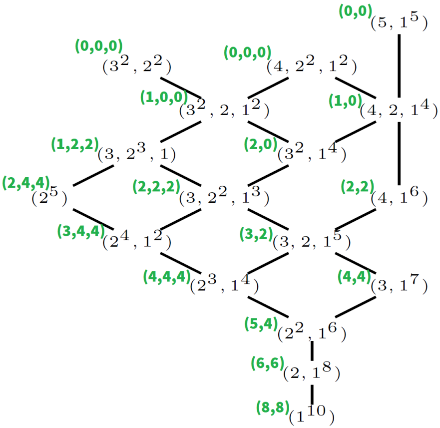
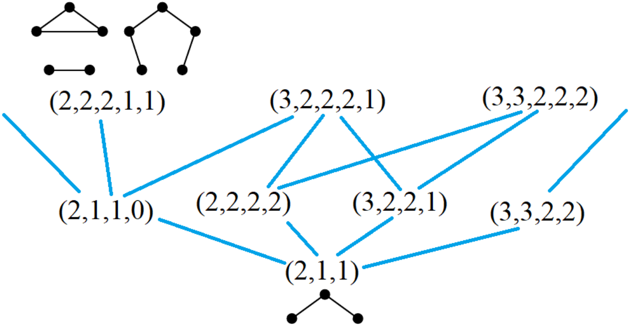


The Rao poset on degree sequences

S.B. Rao (1980): $d \succcurlyeq e$ if
there exist realizations G, H of d, e, respectively, so that G contains H as an **induced subgraph.** We say that d **Rao-contains** e.



Rao's poset links degree sequences and induced subgraphs in (mostly) natural ways... forbidden subgraphs $\sim$ "forbidden sequences"

The Rao poset and EG differences

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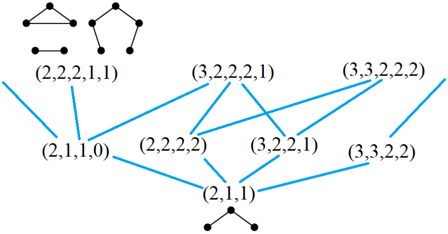

Example

$d = (4,3,3,2,2,2) \succcurlyeq e = (2,2,2,1,1)$

$\Delta(d) = (1,3,2)$ and $\Delta(e) = (2,2,2)$

$e \succcurlyeq f = (2,1,1)$ $\qquad \Delta(f) = (0,0)$

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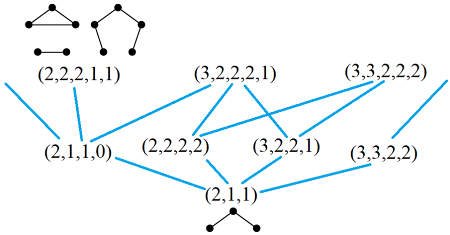

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Theorem [B, 21+]

- If $d \succcurlyeq e$, then $m(d) \geq m(e)$.
- If $d \succcurlyeq e$, then $\Delta_{m(d)}(d) \geq \Delta_{m(e)}(e)$.
- If $d \succcurlyeq e$, then $\max \Delta_j(d) \geq \max \Delta_k(e)$.

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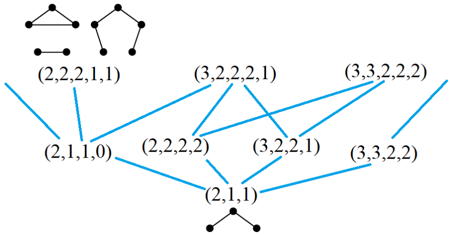

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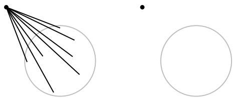
Corollary

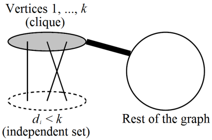
The graphs whose degree sequences d have bounded (m or Δ_m or $\max \Delta_j$) form a hereditary class. (E.g. split, threshold, weakly threshold)

Questions for the future

- Iterative constructions of graphs G for which $\Delta(\deg(G))$ satisfies desired properties?



- A refinement of the Tyshkevich decomposition (where $\Delta_k(d) = 0$ determines breaks) in terms of other values in $\Delta(d)$?



- Other applications of $\Delta(d)$?

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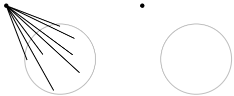

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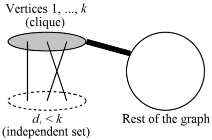

- Other applications of $\Delta(d)$?

Thank you!